Asymptotic probability of majority inversion under a general binomial voting model

Serguei Kaniovski *   Alexander Zaigraev †

February 8, 2021

Abstract

This paper characterizes the limit values of the probability of majority inversion when the number of voters tends to infinity, assuming a binomial model specific to each state, states of different sizes and an arbitrary voting quota. The main asymptotic theorem provides the limit values for most parametrizations of the model. A prominent special case in which the limit cannot be determined using the theorem is the classical binomial model. To address this well-known special case, we provide both an exact and an approximate expression for the inversion probability that applies to any parametrizations of the model.

Key Words: two-tier voting, inversion probability, binomial voting model

1 Introduction

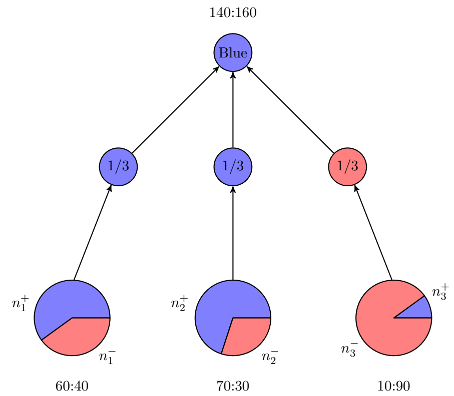
In a two-tier election, voters divided among electoral constituencies (states) cast their votes in favor of one of the two alternatives. In the first stage, a majority in each state is determined. In the second stage, the alternative supported by most of the states wins the election. A *majority inversion* occurs when the majority of the states in the second stage contradicts a nationwide majority of the votes in the first stage. The phenomenon of majority inversion attracts considerable attention in the political science literature and in the theory of voting, as the prospect of majority inversion undermines the democratic legitimacy of an election outcome.

To illustrate the phenomenon of majority inversion, consider the example of three states in Figure 1. The votes are unweighted and the collective decision rule is simple majority. A total turnout of 300 voters is evenly distributed among the three states. Blue wins the election with a majority of states, even though Red would win a hypothetical direct nationwide vote, as the Blue faction $n_1^+ + n_2^+ + n_3^+ = 140$ is smaller then the Red faction $n_1^- + n_2^- + n_3^- = 160$.

*✉Austrian Institute of Economic Research (WIFO),
Arsenal, Object 20, 1030 Vienna, Austria.
E-mail: serguei.kaniovski@wifo.ac.at

†Faculty of Mathematics and Computer Science,
Nicolaus Copernicus University,
Chopin str. 12/18, 87-100 Toruń, Poland.
Email: alzaig@mat.umk.pl

Figure 1: Majority inversion with three states



Blue wins the election by a 2/3 majority of states, while receiving only 140 of 300 votes. This leads to a majority inversion, because Blue is not backed by a nationwide majority.

The probability of majority inversion in an idealized voting model depends on the voting procedure and the stochastic model that determines the voters' choice between the two alternatives. The two stochastic settings dominate the theoretical literature: the Impartial Culture (IC) and the Impartial Anonymous Culture (IAC). The IC setting relies on what we call the *classical binomial model* of voting. In this model, each vote is equally likely to support or oppose an alternative on the ballot, and all votes are independent. The model implies the same level of support for the two alternatives. The probability of majority inversion under the classical binomial model has been studied, for example, in Feix, Lepelley, Merlin and Rouet (2004) and Lepelley, Merlin and Rouet (2011). The main conceptual difference between the IC and the IAC is that the latter admits different levels of support, provided that all conceivable levels of support are equally likely. The level of support is expressed by an absolute number of supporters, which becomes a fraction of supporters asymptotically, as in the pioneering asymptotic analysis by May (1948) and, more recently, in Feix et al. (2004) and Lepelley et al. (2011). The underlying stochastic model assumes that each vote has an equal probability of being in favor of an alternative, and that this probability is drawn from the uniform distribution on a unit-interval. Having assigned the common probability of an affirmative vote, voters cast their votes independently. The model behind the IAC is a special case of a more general compound beta-binomial model, because the continuous uniform distribution from which the common probability parameter is drawn is a special case of a more general beta distribution. Drawing a common probability parameter for the uniform distribution introduces a positive correlation of 1/3 between the affirmative votes, even though the votes are independent, conditioned on the common parameter. The recent paper by De Mouzon, Laurent, Le Breton and Lepelley (2020) shows that the two settings can be combined to obtain an even richer setting in which the IAC assumption holds within each state, but any two votes from different states remain independent, and investigate the bounds on the probability of inversion in this setting. Three simplifying assumptions common to all of the above studies are i) states of equal population size, ii) simple majority rule in the first stage of the two-stage voting procedure, and iii) unweighted votes in the second stage. Each of these assumptions greatly simplifies the combinatorial aspect of the probability calculations.

The IC and IAC dichotomy is rooted in the literature on voting power, where it serves as a probabilistic foundation for the Penrose-Banzhaf and the Shapley-Shubik measures of power (Straffin 1977). According to Morriss (2002) and Felsenthal and Machover (2004), a power measure should reflect *a priori* voting power, or power granted by a set of decision-making rules, rather than the preferences or the behavior of the voters.[1] This perspective favors simple stochastic models that should be as agnostic and neutral as possible with respect to implied voting behavior. The stochastic model considered in this paper introduces a minimal departure from an overly restrictive interpretation of the *a priori* perspective embodied by the classical binomial model. It does so by allowing voters from different states to have different probabilities of supporting the alternative or candidate on the ballot.

In the recent theoretical literature on the probability of majority inversions, exact calculations and estimates of the inversion probability under the two alternative settings often go hand in hand, not least due to the similarity of the analytical methods used to perform them. Yet the IC setting is often considered too contrived and consequently less interesting than the IAC setting because the underlying classical binomial model is extremely rigid and essentially assumes that every voter in every state tosses a fair coin. In this paper we show that asymptotic estimates of the inversion probability can be performed for a richer binomial setting in which different binomial models are assumed for different states. This brings the proposed binomial setting closer to the IAC setting, the difference being that in the former different probability parameters are assumed for each state, whereas in the IAC these parameters are drawn from a uniform distribution. Other generalizations pursued in this paper include relaxing the assumption that states have the same population size and that the decision rule is a simple majority. The present framework thus assumes a binomial model specific to each state, where states can differ in size and the collective decision rule allows for an arbitrary voting quota in the first stage of the two-stage voting procedure.

In the next section, we outline the scope of this paper in terms of the voting procedure and the stochastic voting model considered in it. Before proceeding with the main asymptotic result, we pause to state some nonasymptotic results for the inversion probability in Section 3, which will help to frame the asymptotic analysis in Section 4. A detailed proof of the main theorem can be found in Appendix A. The final section briefly summarizes the main findings.

2 The model

2.1 The voting procedure

The voting procedure is defined by the number of states, the number of voters in each state, the voting weights and the voting quota. We make the following assumptions:

A.1 the number of states s is fixed;

A.2 the number of voters in each state is $n_1, \ldots, n_s$ and their overall number is $N = \sum_{i=1}^{s} n_i$. In the asymptotic analysis, the number of voters in each state and thus their overall number tend to infinity;

A.3 the votes at both stages of the voting procedure are not weighted;

[1] For a debate on the role of preferences in the measurement of voting power, the reader is referred to Napel and Widgrén (2004) and the critique by Braham and Holler (2005a), as well as the reply by Napel and Widgrén (2005) and the rejoinder by Braham and Holler (2005b).

A.4 the voting quota in the first stage is given by $\alpha \in (0, 1)$, where $\alpha = 0.5$ implies a *simple majority* and $\alpha > 0.5$ implies a *qualified majority.* The decision in the second stage is taken by simple majority;

A.5 s is odd, while αN and αn_i for all $i = 1, \ldots, s$ are not integers;

A.6 We do not distinguish between turnout and population, thus ignoring the possibility that not every resident of a state is eligible to vote and that some eligible voters may abstain. Equating turnout and population is permissible if turnout is treated as exogenous. Throughout this paper, we will use the terms "number of voters", "voter turnout" and "population" interchangeably.

The first two assumptions convey the asymptotic nature of our analysis. Assumptions [A.3], [A.5] and [A.6] are fairly common to the existing theoretical studies of two-tier voting models. Real two-tier procedures can feature weighted votes in the second stage, with the weights typically reflecting the size of the population in each state, as is the case with the U.S. Electoral College. The assumption of weighted votes precludes an analytical analysis of the voting model for a large number of states, due to the resulting combinatorial complexity. This is the reason why estimates of the inversion probability and calculations of voting power for general weighted voting models rely on purely numerical methods. Explicit nonasymptotic results for several weighting schemes in the case of three states can be found in Lepelley, Merlin, Rouet and Vidu (2014), Kaniovski and Zaigraev (2018) and Zaigraev and Kaniovski (2020). Empirical models of the US presidential elections can be found in Katz, Gelman and King (2004), Miller (2012), who also documents inversions in presidential elections that occurred in the United States and elsewhere around the world, and Geruso, Spears and Talesara (2019). While we exclude the case of weighted votes [A.3], we still allow the population sizes of states to differ [A.2]. Assumption [A.5] is a common technical assumption that rules out ties at both stages of the voting procedure. This assumption is not particularly restrictive in an asymptotic analysis, but it removes the need to provide a tie-breaking rule. Assumption [A.4] allows us to see the effect of an arbitrary quota on the inversion probability. Most existing theoretical studies assume simple majority rule. In the case of a qualified majority, the quota can be interpreted as the level of support needed to bring about a change in the status quo, such as the adoption of a policy or a constitutional amendment in a nationwide referendum.

2.2 The binomial model

The stochastic model assumes that each voter either supports or opposes the alternative on the ballot. Let $n_i^+ \in [0, n_i]$ be the number of *proponents* and $n_i^- = n_i - n_i^+$ the number of *opponents* in state i, then

A.7 all votes in the country are *stochastically independent* binary random variables,

A.8 the number of proponents n_i^+ follows a *binomial distribution* $\mathcal{B}(n_i, p_i)$, whose probability p_i is specific to the state i.

Assumption [A.8] specifies the probability distribution of the sizes of factions $n_i^+ \sim \mathcal{B}(n_i, p_i)$ and $n_i^- \sim \mathcal{B}(n_i, 1 - p_i)$ as factions who support the same alternative in a given state. The total sizes of the two factions $N^+ = \sum_{i=1}^s n_i^+$ and $N^- = \sum_{i=1}^s n_i^-$ comprise the total population $N = N^- + N^+$. At the level of individual votes, [A.7] and [A.8] are equivalent to all affirmative votes in state i having the same probability and all votes in the country being independent. The assumption of states having different probabilities p_i generalizes the classical binomial model of voting with $p = 0.5$ assumed for each state.

Before proceeding further, we digress briefly on the equivalence of the above binomial model and a Poisson model for a fixed population, with the assumption of a fixed population being an essential prerequisite for the following consideration. Let n^+ and n^- be independent Poisson random variables with parameters λ^+ and λ^-, respectively.

Remark. *The conditional probability* $P(n^+ = k \mid n^+ + n^- = n)$ *for* $k = 0, \ldots, n$ *is binomial*

$$P(n^+ = k \mid n^+ + n^- = n) = \binom{n}{k} \left(\frac{\lambda^+}{\lambda^+ + \lambda^-} \right)^k \left(\frac{\lambda^-}{\lambda^+ + \lambda^-} \right)^{n-k}.$$

To prove this well-known fact, note that

$$P(n^+ = k \mid n^+ + n^- = n) = \frac{P(n^+ = k, n^+ + n^- = n)}{P(n^+ + n^- = n)}.$$

Now, $P(n^+ = k, n^+ + n^- = n) = P(n^+ = k, n^- = n - k)$, and by independence of n^+ and n^-

$$P(n^+ = k, n^- = n - k) = P(n^+ = k)P(n^- = n - k) = \frac{(\lambda^+)^k (\lambda^-)^{n-k}}{k!(n-k)!} e^{-(\lambda^+ + \lambda^-)}.$$

Dividing this probability by $P(n^+ + n^- = n) = \frac{(\lambda^+ + \lambda^-)^n}{n!} e^{-(\lambda^+ + \lambda^-)}$ yields

$$P(n^+ = k \mid n^+ + n^- = n) = \binom{n}{k} \left(\frac{\lambda^+}{\lambda^+ + \lambda^-} \right)^k \left(\frac{\lambda^-}{\lambda^+ + \lambda^-} \right)^{n-k}.$$

The above comment highlights another aspect of the generality of the chosen setting, but this line of argument is not pursued further.

2.3 Probability of majority inversion

The probability of majority inversion is the focus of this paper. Majority inversions occur when the nationwide majority in the first stage does not match the majority of the state in the second stage. Let P_1 be the probability that proponents have a nationwide majority of votes but not a majority of states, and P_2 be the probability that opponents have a nationwide majority of votes but not a majority of states. Since the two scenarios are mutually exclusive events, the inversion probability is equal to the sum $P = P_1 + P_2$, where

$$\begin{aligned} P_1 &= P(N^+ > N\alpha, \text{ at least } (s+1)/2 \text{ of } \{n_i^+ - n_i\alpha\} \text{ are negative}), \\ P_2 &= P(N^+ < N\alpha, \text{ at least } (s+1)/2 \text{ of } \{n_i^+ - n_i\alpha\} \text{ are positive}). \end{aligned}$$

The main result allows us to determine the limit value of the inversion probability for most, but not all, parametrizations. A parametrization of an election scenario comprises: a vector of limit population shares of the states $\mathbf{c}$, a vector of binomial probability parameters $\mathbf{p}$ and a voting quota $\alpha \in (0, 1)$. The main discriminatory quantity with respect to the limit value

$$\lim_{n_1 \to \infty, \ldots, n_s \to \infty} P(\mathbf{p}, \alpha)$$

is the scalar product $\langle \mathbf{c}, \mathbf{p} \rangle$, leading to two principal cases $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$ and $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$. The theorem provides a complete characterization of the limit values for the general case $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$. For the special case $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$, the theorem can only confirm that the limit value is positive, provided that the total population is not asymptotically concentrated in a single state. Otherwise, the limit value can be zero. To the best of our knowledge, this result has so far only been established for the classical binomial model ($p_1, \dots, p_s = 0.5$), states of equal size ($c_1, \dots, c_s = 1/s$) and simple majority rule ($\alpha = 0.5$) by Feix et al. (2004), who provide exact limit values for $s = 3, 4, 5$, as well as simulated limit values for $s > 5$. In a follow-up paper, Lepelley et al. (2011) present an approximate expression for the limit value of the inversion probability when n and s are large, retaining the assumption of equal size and simple majority rule. The simulated limit value for many voters and states published by Feix et al. (2004) has been confirmed analytically by Kikuchi (2016). The case $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$ does not seem to be amenable to analytic methods in general, but for a small number of voters and states the inversion probability can be calculated using an exact formula provided in the next section.

3 Nonasymptotic probability of majority inversion

For a pair of integers a, b, let $\binom{a}{b} = a!/[b!(a-b)!]$ be the binomial coefficient, if $a \geqslant b \geqslant 0$, and $\binom{a}{b} = 0$, otherwise. The following additive decomposition of the inversion probability $P(\mathbf{p}, \alpha)$ can be useful for obtaining theoretical results and in computations:

$$P(\mathbf{p}, \alpha) = \sum_{k=(s+1)/2}^{s-1} \Big[\sum P_1(i_1 \dots, i_k) + \sum P_2(i_1, \dots, i_k) \Big], \tag{1}$$

where for a fixed k each sum has $\binom{s}{k}$ summands and

$$P_1(i_1, \dots, i_k) = P\Big(N^+ > N\alpha, n_{i_1}^+ < n_{i_1}\alpha, \dots, n_{i_k}^+ < n_{i_k}\alpha, n_{i_{k+1}}^+ > n_{i_{k+1}}\alpha, \dots, n_{i_s}^+ > n_{i_s}\alpha\Big),$$
$$P_2(i_1, \dots, i_k) = P\Big(N^+ < N\alpha, n_{i_1}^+ > n_{i_1}\alpha, \dots, n_{i_k}^+ > n_{i_k}\alpha, n_{i_{k+1}}^+ < n_{i_{k+1}}\alpha, \dots, n_{i_s}^+ < n_{i_s}\alpha\Big).$$

In P_1, $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant s$ indicate the states in which the opponents win, whereas in P_2 they indicate the states in which the proponents win.

The next expression for $P(\mathbf{p}, \alpha)$ follows directly from Assumption [A.8]:

Exact Formula. *The probability of majority inversion $P(\mathbf{p}, \alpha) = P_1(\mathbf{p}, \alpha) + P_2(\mathbf{p}, \alpha)$,*

$$P_1(\mathbf{p}, \alpha) = \sum_{k=(s+1)/2}^{s-1} \sum_{t=[N\alpha]+1}^{t^*(k)} \sum_{\{k_i\} \in A_{t,k}} \prod_{j=1}^{s} \binom{n_j}{k_j} p_j^{k_j} (1-p_j)^{n_j - k_j},$$
$$P_2(\mathbf{p}, \alpha) = \sum_{k=(s+1)/2}^{s-1} \sum_{t=t_*(k)}^{[N\alpha]} \sum_{\{k_i\} \in A'_{t,k}} \prod_{j=1}^{s} \binom{n_j}{k_j} p_j^{k_j} (1-p_j)^{n_j - k_j},$$

where

$$A_{t,k} = \{k_1 + k_2 + \dots + k_s = t, k \textit{ integers in } [0, n_i\alpha), s-k \textit{ integers in } (n_i\alpha, n_i]\},$$
$$A'_{t,k} = \{k_1 + k_2 + \dots + k_s = t, k \textit{ integers in } (n_i\alpha, n_i], s-k \textit{ integers in } [0, n_i\alpha)\},$$

and, for $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s$,

$$t^*(k) = [n_1\alpha] + \cdots + [n_k\alpha] + n_{k+1} + \cdots + n_s,$$
$$t_*(k) = [n_1\alpha] + \cdots + [n_k\alpha] + k.$$

The complexity of computing the inversion probability using the above formula results from the need to determine all elements of the sets $A_{t,k}$ and $A'_{t,k}$ as the integers lying on the hyperplanes defined by $k_1 + k_2 + \cdots + k_s = t$. The summation bounds $t_*(k)$ and $t^*(k)$ reduce the number of summands in $P_1(\mathbf{p}, \alpha)$ and $P_2(\mathbf{p}, \alpha)$ by omitting those values of $\{k_i\}$ for which inversions cannot occur.

The expression for the inversion probability possesses the following symmetry:

Corollary 1. $P_1(\mathbf{p}, \alpha) = P_2(\mathbf{1} - \mathbf{p}, 1 - \alpha)$ *and* $P_1(\mathbf{1} - \mathbf{p}, 1 - \alpha) = P_2(\mathbf{p}, \alpha)$ *and, consequently,* $P(\mathbf{p}, \alpha) = P(\mathbf{1} - \mathbf{p}, 1 - \alpha)$.

Indeed, if n_i^+ has the distribution $\mathcal{B}(n_i, p_i)$, then $n_i - n_i^+$ has the distribution $\mathcal{B}(n_i, 1 - p_i)$. To prove $P_2(\mathbf{p}, \alpha) = P_1(\mathbf{1} - \mathbf{p}, 1 - \alpha)$, note that

$$\begin{aligned} P_2(\mathbf{p}, \alpha) &= P(N^+ < N\alpha, \text{ at least } (s+1)/2 \text{ of } \{n_i^+ - n_i\alpha\} \text{ are positive}),\\ &= P(N^+ - N < N\alpha - N, \text{ at least } (s+1)/2 \text{ of } \{n_i^+ - n_i + n_i(1-\alpha)\} \text{ are positive}),\\ &= P(N - N^+ > N(1-\alpha), \text{ at least } (s+1)/2 \text{ of } \{(n_i - n_i^+) - n_i(1-\alpha)\} \text{ are negative}),\\ &= P_1(\mathbf{1} - \mathbf{p}, 1 - \alpha). \end{aligned}$$

The equality $P_1(\mathbf{p}, \alpha) = P_2(\mathbf{1} - \mathbf{p}, 1 - \alpha)$ can be proven in a similar manner.

For the classical binomial model $P(0.5, 0.5) = 2P_1(0.5, 0.5) = 2P_2(0.5, 0.5)$. This symmetry can somewhat simplify the calculation of the inversion probability. The computational complexity under the classical binomial model can be further reduced by assuming that all states have the same number of voters.

Corollary 2. *For* $n_i^+ \sim \mathcal{B}(n, 0.5)$ *for all* $i = 1, \ldots, s$ *and* $\alpha = 0.5$, *we have*

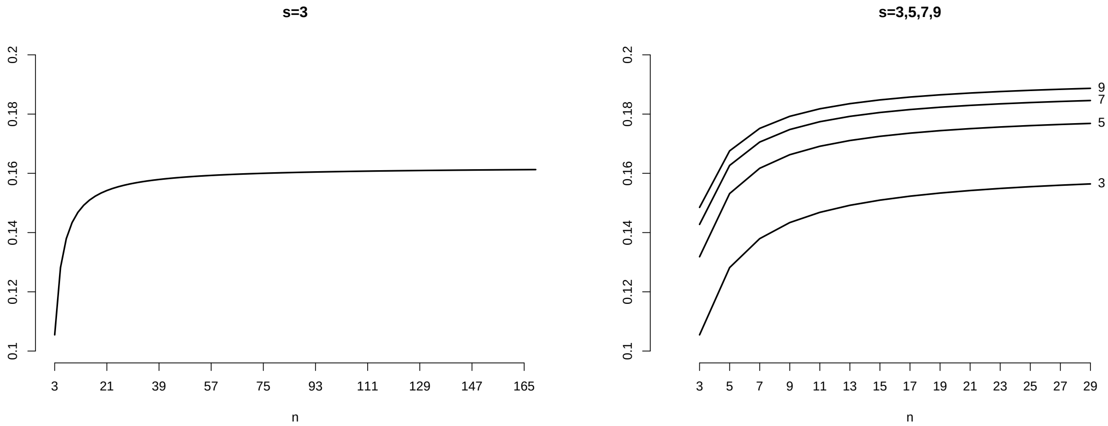
$$P(0.5, 0.5) = 2^{1-sn} \sum_{k=(s+1)/2}^{s-1} \binom{s}{k} \sum_{t=(sn+1)/2}^{sn-k(n+1)/2} \sum_{\{k_i\} \in A_{t,k}} \binom{n}{k_1}\binom{n}{k_2} \cdot \ldots \cdot \binom{n}{k_s},$$

where

$$A_{t,k} = \{k_1 + \cdots + k_s = t, \textit{ the first } k \textit{ integers in } [0, (n-1)/2], s - k \textit{ integers in } [(n+1)/2, n]\}.$$

Figure 2 illustrates the inversion probability given in Corollary 2. The probability appears to converge as $n \to \infty$ for a fixed s (left panel), for $s \to \infty$ for a fixed n, and for $n \to \infty$ and $s \to \infty$ (right panel). The scenario $n \to \infty$ for a fixed s is the most relevant of the three asymptotic scenarios because it can serve as an approximation of real-world electoral systems with a large electorate residing in a given number of states. In the next section we analyze the set of feasible limit values for the inversion probability under a more general binomial model in which p is specific to each state and the states can differ in size.

Figure 2: Inversion probability for equally-sized states



The figure illustrates the inversion probability under the classical binomial model, simple majority rule and states of equal size given in Corollary 2. The probability appears to converge as the size of a state $n \to \infty$ for a fixed number of states s, for $s \to \infty$ for a fixed n, as well as for $n \to \infty$ and $s \to \infty$.

4 Asymptotic probability of majority inversion

Let c_i be the limit population share of state i in the total population, $i = 1, \dots, s$, such that $\sum_{i=1}^{s} c_i = 1$. The limit shares need not be equal and can include the borderline case of a single state absorbing the entire population by allowing any one of the shares to be equal to 1. Next, we introduce $\mathbf{m}$, the vector of *election margins* in each state:

Definition. *For a vector* $\mathbf{m}$ *with elements* $m_i = \alpha - p_i, i = 1, \dots, s$, *denote the number of negative, zero and positive elements in* $\mathbf{m}$ *by* m^-, m^0 *and* m^+, *respectively.*

Since we are dealing with a partition of $\mathbf{m}$, the cardinalities of the three subsets $m^- + m^0 + m^+ = s$, where s is the number of elements in $\mathbf{m}$, or the number of states. The vector of margins will play a prominent role in the following analysis. We will say that (the election of) state i is *close* if the i'th element of $\mathbf{m}$ is equal to 0, that is $p_i = \alpha$.

Asymptotic Theorem. *For the limit of the probability of inversion it holds that:*

a) if $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$, *then*

$$\lim_{n_1 \to \infty, \dots, n_s \to \infty} P(\mathbf{p}, \alpha) = \begin{cases} 2^{-m^0} \sum_{k=0}^{(s-1)/2 - m^+} \binom{m^0}{k}, \ \textit{for } 1 \leqslant m^+ \leqslant (s-1)/2 \\ 0, \ \textit{for } (s+1)/2 \leqslant m^+ \leqslant s; \end{cases} \tag{2}$$

b) if $\langle \mathbf{c}, \mathbf{p} \rangle > \alpha$, *then*

$$\lim_{n_1 \to \infty, \dots, n_s \to \infty} P(\mathbf{p}, \alpha) = \begin{cases} 2^{-m^0} \sum_{k=0}^{(s-1)/2 - m^-} \binom{m^0}{k}, \ \textit{for } 1 \leqslant m^- \leqslant (s-1)/2 \\ 0, \ \textit{for } (s+1)/2 \leqslant m^- \leqslant s; \end{cases} \tag{3}$$

c) if $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$, *then the limit is positive, unless not all* $\{p_i\}$ *are equal and one of* $\{c_i\}$ *is equal to 1, in which case the limit can be 0.*

A proof of the theorem is given in Appendix A.

Asymptotic Theorem gives the limit of the inversion probability for all cases, except for $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$. Let us discuss the two cases $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$ and $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$ in some detail.

4.1 Limit values when $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$

Determining the limit in the general case requires checking several conditions. The first step is to verify if the probability vanishes in the limit. Depending on whether $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$ or $\langle \mathbf{c}, \mathbf{p} \rangle > \alpha$, this occurs if $m^+ \geqslant (s+1)/2$ or $m^- \geqslant (s+1)/2$. In view of the definition of $\mathbf{m}$ as a vector with elements $\alpha - p_i$, this happens because the nationwide majority and the majority of states agree. For all other parametrizations, the inversion probability converges to a positive value given by either (2) or (3) that can be found by plugging m^0 and, depending on whether $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$ or $\langle \mathbf{c}, \mathbf{p} \rangle > \alpha$, either m^+ or m^-.

Interestingly, according to Equations (2)-(3), the limits can be collected in a table that depends only on the number of states s; see such tables for $s = 3, 5, 9$ below.

$s = 3$

	$j=1$
$m^0 = 0$	1
$m^0 = 1$	1/2
$m^0 = 2$	1/4

$s = 5$

	$j=1$	$j=2$
$m^0 = 0$	1	1
$m^0 = 1$	1	1/2
$m^0 = 2$	3/4	1/4
$m^0 = 3$	1/2	1/8
$m^0 = 4$	5/16	•

$s = 9$

	$j=1$	$\underline{j=2}$	$j=3$	$j=4$
$m^0 = 0$	1	1	1	1
$m^0 = 1$	1	1	1	1/2
$m^0 = 2$	1	1	3/4	1/4
$\underline{m^0 = 3}$	1	$\underline{7/8}$	1/2	1/8
$m^0 = 4$	15/16	11/16	5/16	1/16
$m^0 = 5$	13/16	1/2	3/16	1/32
$m^0 = 6$	21/32	11/32	7/64	•
$m^0 = 7$	1/2	29/128	•	•
$m^0 = 8$	93/256	•	•	•

The rows run through the relevant values for the number of states m^0 that are close, while the columns run through the relevant values of $j = m^+$ or $j = m^-$, that is the number of states that are not close and match the nationwide majority. The dimensions of the table are given by the number of elements in the sequences $0, \ldots, s-j$ and $1, \ldots, (s-1)/2$, with entries outside of these ranges marked by $\bullet$.

Let us apply the right-most table to the below parametrization in the case of nine states ($s = 9$) and simple majority rule $\alpha = 0.5$:

State	1	2	3	4	5	6	7	8	9
Asymptotic shares $\mathbf{c}$	0.05	0.05	0.05	0.1	0.1	0.15	0.15	0.15	0.2
Probabilities $\mathbf{p}$	0.5	0.2	0.5	0.3	0.4	0.9	0.3	0.5	0.7
Margins $\mathbf{m}$	0.0	0.3	0.0	0.2	0.1	-0.4	0.2	0.0	-0.2
$m^- = 2$						✓			✓
$m^0 = 3$	✓		✓					✓	
$m^+ = 4$		✓		✓	✓		✓		

In this example, $\langle \mathbf{c}, \mathbf{p} \rangle = 0.525$, so that $\langle \mathbf{c}, \mathbf{p} \rangle > \alpha$. The inversion probability is positive, as $m^- = 2$ is smaller than $(s+1)/2 = 5$. The number of close states is given by $m^0 = 3$. Equation (3) yields the (underlined) limit value of 7/8. Let us now slightly change the parametrization by lowering the probability for the fifth state from $p_5 = 0.4$ to $p_5 = 0.1$. In this case $\langle \mathbf{c}, \mathbf{p} \rangle = 0.495$, so that $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$. To verify whether the probability is positive, we now have to check if m^+ is smaller than $(s+1)/2$, as is the case because $m^+ = 4$ and $(s+1)/2 = 5$.

The number of close states, $m^0 = 3$, is the same as in the previous example. Equation (2) yields a new limit value of 1/8.

To round up the discussion of the general case, let us revisit the tables for $s = 3, 5, 9$. Each limit in the first row is equal to 1, meaning that the absence of close states $m^0 = 0$ implies a sure inversion, under the conditions for which the table holds. This, in turn, implies that in the general case $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$, the absence of close states is a *sufficient* condition for a limit value of 0 or 1. A *necessary* and *sufficient* condition for the limit value to be *different* from 0 or 1 is furnished by a pair of inequalities $m^- \leqslant (s-1)/2$ and $m^+ \leqslant (s-1)/2$. If all states have a common p, the inversion probability vanishes asymptotically. This result does not depend on the shares, since the condition $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$ simplifies to $p \neq \alpha$ in view of $\sum_{i=1}^{s} c_i = 1$.

4.2 Limit values when $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$

Let us now turn to this important but special case. Here, the Asymptotic Theorem does not yield a limit value for the inversion probability. The theorem only establishes the positivity of the limit for all cases where the entire population is not asymptotically concentrated in a single state. In the unrealistic scenario of the entire population concentrating in a single state, the limit of the inverse probability is smaller than 1 and can be equal to 0, if $\alpha - p_i = 0$ in that state ($c_i = 1$). For all other parametrizations such that $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$, the Exact Formula and its Corollary 2 offers a means of approximating the limit value numerically. But applying these nonasymptotic results is computationally intensive, so that stochastic approximation may be a more appealing alternative to simulating the limit value.

The case $p = \alpha$ emerges as the only nontrivial asymptotic case with respect to the inversion probability in a binomial setting with a common p, a result that was anticipated in Feix et al. (2004). For the classical binomial model with equally large states and simple majority rule ($p = \alpha = 0.5$), Feix et al. (2004) provide the exact limit values for $s = 3, 4, 5$, as well as a numerically simulated limit value for $s > 5$. An approximate expression for large n and s is given in Lepelley et al. (2011). The limit value for the case of infinitely many states obtained by simulation in Feix et al. (2004) has been confirmed analytically by Kikuchi (2016), who also provides the expression:

$$\frac{1}{2} - \frac{1}{\pi} \cdot \arctan\left(\sqrt{\frac{2}{\pi - 2}}\right) \approx 0.2059524.$$

This value is valid for any $p = \alpha$, not only for $p = \alpha = 0.5$.

4.3 An approximation for the case $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$

The following approximation of the probability of majority inversion generalizes the similar result obtained by Lepelley et al. (2011) for states of equal population size and simple majority rule. The point of departure for the following discussion is their approximation to the inversion probability for a large number of states. The main idea behind the approximation is that for independently and identically distributed votes the distribution of a normalized margin of victory in each state tends to a normal law. The key parameters in their analysis are the probability p_A that alternative A enjoys a simple majority of votes in any given state, where all states have the same number of voters, n, and the number of states s_A that support A. Similarly, $1 - p_A$ is the probability that alternative B wins in any given state, with $s - s_A$ being the number of states that support B. Further quantities of interest include:

$$m = s_A m_A + (s - s_A) m_B,$$
$$\sigma^2 = s_A \sigma_A^2 + (s - s_A) \sigma_B^2,$$

where m_A and σ_A are the expected value and the standard deviation or a random variable that approximates the margin of victory of alternative A in those states in which A wins. Similarly, m_B and σ_B are the expected value and the standard deviation or a random variable that approximates the margin of defeat of alternative A in those states in which B wins (for details, see the exposition on p. 32 in Lepelley et al. (2011)).

The approximate inversion probability for large s according to Equation (32) of Lepelley et al. (2011) reads

$$\lim_{n\to\infty} P(n,s) \approx \sum_{s_A=(s+1)/2}^{s-1} \binom{s}{s_A} p_A^{s_A}(1-p_A)^{s-s_A} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big)dx, \tag{4}$$
$$+ \sum_{s_A=1}^{(s-1)/2} \binom{s}{s_A} p_A^{s_A}(1-p_A)^{s-s_A} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big)dx. \tag{5}$$

As the number of voters n_i in state i increases, the distribution of $n_i^+ - Nc_ip_i$ is, approximately, a normal distribution $\mathcal{N}(0, N\sigma_i^2)$ with $\sigma_i^2 = c_ip_i(1-p_i)$, for $i = 1, \ldots, s$. Therefore, if A wins in state i, then we can approximate m_A and σ_A in this state as $m_A^i = \sqrt{2/\pi}\sigma_i$ and $\sigma_A^i = \sigma_i\sqrt{1-2/\pi}$. Similarly, $m_B^i = -\sqrt{2/\pi}\sigma_i$ and $\sigma_B^i = \sigma_A^i = \sigma_i\sqrt{1-2/\pi}$.

Consider the first sum (4) and the vector $\mathbf{m}$ with elements $\alpha - p_i$. For a fixed $s_A = k$, there are $\binom{s}{k}$ summands representing the set of feasible victory margins when A wins in k states and loses in $s-k$ states. But some summands in (4) will vanish, because $p_A \to 1$ for states with $m_i < 0$ and $p_A \to 0$ for states with $m_i > 0$. For nonzero summands, the states with $m_i < 0$ are among those k states where A wins, the states with $m_i > 0$ are among those $s-k$ states where A loses.

The calculation for $s_A = k$, where $(s+1)/2 \leqslant k \leqslant s-1$, is the same as the calculation of the number of probabilities satisfying condition (6) in the proof of the Asymptotic Theorem in Appendix A. The first sum (4) equals

$$2^{-m^0} \sum_{k=(s+1)/2}^{s-1} \binom{m^0}{s-k-m^+} \Phi(-\zeta_k),$$

where $\Phi(\cdot)$ is the distribution function corresponding to $\mathcal{N}(0,1)$ and

$$\zeta_k = \frac{m_k}{\sigma} = \sqrt{\frac{2}{\pi-2}} \cdot \frac{\sum_{i=1}^{k}\sigma_i - \sum_{r=1}^{s-k}\sigma_{k+r}}{\sqrt{\sum_{i=1}^{s}\sigma_i^2}}.$$

Similarly, one can calculate the second sum (5) as:

$$2^{-m^0} \sum_{k=1}^{(s-1)/2} \binom{m^0}{k-m^-} \Phi(\zeta_k).$$

In total, we obtain the following approximation of the inversion probability for large s when $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$:

$$\lim_{n_1 \to \infty, \ldots, n_s \to \infty} P(\mathbf{p}, \alpha) \approx 2^{-m^0} \sum_{k=(s+1)/2}^{s-1} \binom{m^0}{s-k-m^+} \Phi(-\zeta_k) + 2^{-m^0} \sum_{k=1}^{(s-1)/2} \binom{m^0}{k-m^-} \Phi(\zeta_k).$$

The above approximation is significantly less computationally expensive than the Exact Formula in Section 3 and should be reasonably accurate even for small values of s.

5 Summary

This paper provides the limit values of the probability of majority inversion for most but not all parametrizations of a state-specific binomial voting model when the number of voters in each state tends to infinity but the number of states is fixed. This asymptotic assumption is intended as an approximation to real two-stage electoral systems with a large electorate residing in a small number of states. The main result provides the limit values for the inversion probability for most but not all parametrizations of the model. The most prominent special case in which the limit cannot be determined using the theorem is the classical binomial model of voting. This special case has been extensively studied in the existing literature, to which we contribute a new exact expression for the inversion probability that holds for any parametrization of the model, as well as an approximation along the lines suggested by Lepelley et al. (2011).

On a closing note, by considering a binomial model specific to each state with different-sized states and an arbitrary voting quota, the results provided in this paper complement the exact and approximate results available in the theoretical literature on the probability of majority inversions for the classical binomial model with equally-sized states and simple majority rule. We believe that a comprehensive characterization of the limits leads to a better understanding of the implications of a binomial setting in the context of majority inversions and makes the binomial setting a more attractive analytical assumption than the special case of the classical binomial model may have suggested so far.

References

Braham, M. and Holler, M.: 2005a, Power and preferences again. A reply to Napel and Widgrén, *Journal of Theoretical Politics* **17**, 389–395.

Braham, M. and Holler, M.: 2005b, The impossibility of a preference-based power index, *Journal of Theoretical Politics* **17**, 137–157.

De Mouzon, O., Laurent, T., Le Breton, M. and Lepelley, D.: 2020, The theoretical Shapley–Shubik probability of an election inversion in a toy symmetric version of the US presidential electoral system, *Social Choice and Welfare* **54**, 363–395.

Feix, M. R., Lepelley, D., Merlin, V. R. and Rouet, J.-L.: 2004, The probability of conflicts in a U.S. presidential type election, *Economic Theory* **23**, 227–257.

Felsenthal, D. S. and Machover, M.: 2004, A priori voting power: What is it all about?, *Political Studies Review* **2**, 1–23.

Geruso, M., Spears, D. and Talesara, I.: 2019, Inversions in US presidential elections, 1836-2016. Working Paper, The University of Texas at Austin.

Kaniovski, S. and Zaigraev, A.: 2018, The probability of majority inversion in a two-stage voting system with three states, *Theory and Decision* **84**, 525–546.

Katz, J. N., Gelman, A. and King, G.: 2004, Empirically evaluating the electoral college, *in* A. N. Crigler, M. R. Just and E. J. McCaffery (eds), *In Rethinking the Vote: The Politics and Prospects of American Election Reform*, Oxford University Press, pp. 75–88.

Kikuchi, K.: 2016, The likelihood of majority inversion in an indirect voting system. Working Paper, Waseda University - School of Political Science and Economics.

Lepelley, D., Merlin, V. R. and Rouet, J.-L.: 2011, Three ways to compute accurately the probability of the referendum paradox, *Mathematical Social Sciences* **62**, 28–33.

Lepelley, D., Merlin, V. R., Rouet, J.-L. and Vidu, L.: 2014, Referendum paradox in a federal union with unequal populations: the three state case, *Economics Bulletin* **34**, 2201–2207.

May, K.: 1948, Probabilities of certain election results, *The American Mathematical Monthly* **55**, 203–209.

Miller, N. R.: 2012, Election inversions by the U.S. Electoral College, *in* D. S. Felsenthal and M. Machover (eds), *Electoral Systems. Studies in Choice and Welfare*, Springer-Verlag Berlin Heidelberg, pp. 93–127.

Morriss, P.: 2002, *Power: A philosophical analysis*, Manchester University Press.

Napel, S. and Widgrén, M.: 2004, Power measurement as sensitivity analysis: a unified approach, *Journal of Theoretical Politics* **16**, 517–538.

Napel, S. and Widgrén, M.: 2005, The possibility of a preference-based power index, *Journal of Theoretical Politics* **17**, 377–387.

Straffin, P. D.: 1977, Homogeneity, independence and power indices, *Public Choice* **30**, 107–118.

Zaigraev, A. and Kaniovski, S.: 2020, Inequality in population weights, majority threshold and the inversion probability in the case of three states, *Operations Research Letters* **48**, 732–736.

A Proof of Asymptotic Theorem

Consider the decomposition of the probability of majority inversion (1), whose summands can be expressed in terms of $\eta_i = \frac{n_i^+ - n_i p_i}{\sqrt{n_i p_i(1-p_i)}}$ and $\mu_i = \frac{\alpha - p_i}{\sqrt{p_i(1-p_i)}}$ for $i = 1, \dots, s$ as

$$P_1(i_1, \dots, i_k) = P\Big(\sum_{i=1}^{s} \sqrt{n_i p_i(1-p_i)}\eta_i > N\alpha - \sum_{i=1}^{s} n_i p_i,$$

$$\eta_{i_1} < \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} < \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} > \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} > \sqrt{n_{i_s}}\mu_{i_s}\Big),$$

$$P_2(i_1, \dots, i_k) = P\Big(\sum_{i=1}^{s} \sqrt{n_i p_i(1-p_i)}\eta_i < N\alpha - \sum_{i=1}^{s} n_i p_i,$$

$$\eta_{i_1} > \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} > \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} < \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} < \sqrt{n_{i_s}}\mu_{i_s}\Big).$$

Letting the number of voters in each state tend to infinity $n_1 \to \infty, \dots, n_s \to \infty$ and their shares in the total population converge to constant values $n_1/N \to c_1, \dots, n_s/N \to c_s$, the above probabilities can be approximated as

$$P_1(i_1, \dots, i_k) \approx P\Big(\sum_{i=1}^{s} \sqrt{c_i p_i(1-p_i)}\eta_i > \sqrt{N}(\alpha - \langle \mathbf{c}, \mathbf{p} \rangle),$$

$$\eta_{i_1} < \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} < \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} > \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} > \sqrt{n_{i_s}}\mu_{i_s}\Big),$$

$$P_2(i_1, \dots, i_k) \approx P\Big(\sum_{i=1}^{s} \sqrt{c_i p_i(1-p_i)}\eta_i < \sqrt{N}(\alpha - \langle \mathbf{c}, \mathbf{p} \rangle),$$

$$\eta_{i_1} > \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} > \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} < \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} < \sqrt{n_{i_s}}\mu_{i_s}\Big).$$

Case $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$:

If a) $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$, then, due to the inequality $\sum_{i=1}^{s} \sqrt{c_i p_i(1-p_i)}\eta_i > \sqrt{N}(\alpha - \langle \mathbf{c}, \mathbf{p} \rangle)$, the probability $P_1(i_1, \dots, i_k) \to 0$ for any $(s+1)/2 \leqslant k \leqslant s-1$ and any $(i_1, \dots, i_k)$. Therefore, $\sum_{k=(s+1)/2}^{s-1} \sum P_1(i_1, \dots, i_k) \to 0$, but

$$P_2(i_1, \dots, i_k) \approx P\Big(\eta_{i_1} > \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} > \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} < \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} < \sqrt{n_{i_s}}\mu_{i_s}\Big)$$

$$\approx \prod_{j=1}^{k} \Big(1 - \Phi(\sqrt{n_{i_j}}\mu_{i_j})\Big) \prod_{r=1}^{s-k} \Phi\Big(\sqrt{n_{i_{k+r}}}\mu_{i_{k+r}}\Big).$$

If b) $\langle \mathbf{c}, \mathbf{p} \rangle > \alpha$, then, due to the inequality $\sum_{i=1}^{s} \sqrt{c_i p_i(1-p_i)}\eta_i < \sqrt{N}(\alpha - \langle \mathbf{c}, \mathbf{p} \rangle)$, the probability $P_2(i_1, \dots, i_k) \to 0$ for any $(s+1)/2 \leqslant k \leqslant s-1$ and any $(i_1, \dots, i_k)$. Therefore, $\sum_{k=(s+1)/2}^{s-1} \sum P_2(i_1, \dots, i_k) \to 0$, but

$$P_1(i_1, \dots, i_k) \approx P\Big(\eta_{i_1} < \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} < \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} > \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} > \sqrt{n_{i_s}}\mu_{i_s}\Big)$$

$$\approx \prod_{j=1}^{k} \Phi(\sqrt{n_{i_j}}\mu_{i_j}) \prod_{r=1}^{s-k} \Big(1 - \Phi(\sqrt{n_{i_{k+r}}}\mu_{i_{k+r}})\Big).$$

Since cases a) and b) are symmetrical, we focus on case a).

a) $\langle \mathbf{c}, \mathbf{p} \rangle < \alpha$. Here $m^+ \geqslant 1$, but if $m^+ \geqslant (s+1)/2$, then $P_2(i_1, \dots, i_k) \to 0$ for any $(s+1)/2 \leqslant k \leqslant s-1$ and any $(i_1, \dots, i_k)$. Therefore, the inversion probability (1) tends to 0.

Let $1 \leqslant m^+ \leqslant (s-1)/2$ and assume, without any loss of generality, that $m_1, \dots, m_s$ are sorted in an ascending order.

For $\mu \in \mathbb{R}$, as $n \to \infty$,

$$\Phi(\sqrt{n}\mu) \to \begin{cases} 1, & \text{for } \mu > 0 \\ 0, & \text{for } \mu < 0 \end{cases} \quad \text{and } \Phi(\sqrt{n}\mu) = 1/2 \text{ for } \mu = 0.$$

Therefore, any probability $P_2(i_1, \dots, i_k)$, which does not satisfy the condition:

$$\{1, \dots, m^-\} \subset \{i_1, \dots, i_k\} \text{ and } \{s - m^+ + 1, \dots, s\} \subset \{i_{k+1}, \dots, i_s\} \tag{6}$$

tends to 0, whereas any probability $P_2(i_1, \dots, i_k)$, which satisfies the condition (6), tends to 2^{-m^0}. It remains to calculate the number of probabilities satisfying (6).

First, consider the case $k = (s+1)/2$. For probabilities satisfying (6) there are $(s-1)/2 - m^+$ free indexes among $\{i_{(s+3)/2}, \dots, i_s\}$, so the number of such probabilities is $\binom{m^0}{(s-1)/2 - m^+}$. For $k = (s+3)/2$ there are $(s-3)/2 - m^+$ free indexes among $\{i_{(s+5)/2}, \dots, i_s\}$, so the number of such probabilities is $\binom{m^0}{(s-3)/2 - m^+}$, and so on. Therefore, the total number of such probabilities reads

$$\sum_{k=0}^{(s-1)/2 - m^+} \binom{m^0}{k},$$

which furnishes the proof of (2).

Case c) $\langle \mathbf{c}, \mathbf{p} \rangle = \alpha$:

We have

$$P_1(i_1, \dots, i_k) \approx P\Big(\sum_{i=1}^{s} \sqrt{c_i p_i (1-p_i)}\eta_i > 0,$$
$$\eta_{i_1} < \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} < \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} > \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} > \sqrt{n_{i_s}}\mu_{i_s}\Big),$$
$$P_2(i_1, \dots, i_k) \approx P\Big(\sum_{i=1}^{s} \sqrt{c_i p_i (1-p_i)}\eta_i < 0,$$
$$\eta_{i_1} > \sqrt{n_{i_1}}\mu_{i_1}, \dots, \eta_{i_k} > \sqrt{n_{i_k}}\mu_{i_k}, \eta_{i_{k+1}} < \sqrt{n_{i_{k+1}}}\mu_{i_{k+1}}, \dots, \eta_{i_s} < \sqrt{n_{i_s}}\mu_{i_s}\Big).$$

If not all $\{p_i\}$ are equal, then, approximately, both parts of the inversion probability (1) are not equal to 0 simultaneously if any $c_i < 1$. Compared to the case $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$, due to the additional first inequality in the probability, any $P_2(i_1, \dots, i_k)$ or $P_1(i_1, \dots, i_k)$, which is approximately positive in a) or b), now becomes approximately smaller but remains positive. If one of the population shares $\{c_i\}$ is equal to 1, then some of the probabilities $P_2(i_1, \dots, i_k)$ or $P_1(i_1, \dots, i_k)$, which are approximately positive in the case $\langle \mathbf{c}, \mathbf{p} \rangle \neq \alpha$, vanish, because of a contradiction between the first and the other inequalities in the probability. For example, when $m^- = m^+ = (s-1)/2$ and the entire population resides in a single close state, so that $m^0 = 1$, the probability of majority inversion vanishes in the limit.

If $p_1 = \dots = p_s$, then the limit value for the inversion probability is always positive, since both parts of the inversion probability (1) are positive.