Business cycles and conditional credit-rating migration matrices

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Abstract

To quantify the impact of business cycles on the dynamics of credit ratings, conditional migration matrices and probabilities of the corresponding macroeconomic scenarios are estimated. The approach is tested on a Standard and Poor's (S&P's) dataset that covers the period from 1991 to 2013. The difference between the conditional probabilities and their unconditional counterparts is evaluated. It is the greatest, up to 300%, for contraction periods and downgrading probabilities.

Keywords: business cycle, credit-rating migration, representative agent, conditional migration probability.

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1 Introduction

The traditional approach in the credit-risk literature considers each credit-rating migration as a trajectory of a time-homogeneous Markov chain. The corresponding transition probabilities are referred to as historical. See among others Jarrow et al. (1997), Gupton et al. (1997), Altman (1998) and Trueck and Rachev (2009) for a theoretical background of this approach and its implementation in the practice of credit-risk analysis.

Seeking to improve the financial sector's resilience to stress scenarios, several authors suggested models in which migration probabilities are adjusted according to macroeconomic conditions. Structural breaks, multi-regime transition matrices and hidden Markov chains are some of the technical means used for modeling and estimating conditional migration probabilities in Bangia et al. (2002), McNeil and Wendin (2007), Korolkiewicz and Elliott (2008), Frydman and Schuermann (2008), Stefanescu et al. (2009), Fei et al. (2012), and Xing et al. (2012).

The Basel III Accord, issued in 2011 by the Basel Committee on Banking Supervision (BCBS), stimulates developing "in-house" risk models that enhance the ability of banks to foresee and absorb shocks caused by financial and economic stress. An important input of such models is default probabilities. They have to be calibrated for the specific portfolio of assets held by a bank. The knowledge of distributions of macroeconomic outcomes and probabilities characterizing their dynamics can further facilitate tailoring risk models to the needs of a bank. In particular, the set of credit events determining riskiness of the portfolio in questions can be narrowed by ignoring unlikely outcomes for the composition of assets in this portfolio. The Banking Supervision Accords typically set 0.1% as the upper limit for the probability of undesirable credit events.

Our contribution to the literature on credit-risk comprises models for estimating conditional migration matrices and probabilities of the corresponding macroeconomic outcomes. This is a toolkit for adjusting the historical migration matrices, a key element of the CreditMetrics approach, to the portfolio in hand. The paper is structured as follows. Section 2 presents the modeling approach – coupling schemes. Main assumptions are given in Section 3. First, a setting without macroeconomic dynamics is considered. Then a model based on a generalization of the regime-switching matrix is presented. The discussion focuses on the corresponding conditional migration distributions. The inputs used for testing the suggested estimators are characterized in Section 4 and Appendix 1. Section 5 contains numerical results and their conceptual interpretation. Section 6 concludes. Appendix 2 describes the maximum likelihood optimization problems used in the estimation.

2 Modeling of credit-rating migrations with coupling schemes

Given a historical transition matrix P, Kaniovski and Pflug (2007) suggested a coupling scheme for rendering individual credit-rating migrations dependent. The term coupling scheme means that a migration is split into an idiosyncratic and a common component, each governed by the same migration matrix P. A purely idiosyncratic dynamics is used within the CreditMetrics approach. The common component induces probabilistic dependence among individual migrations.

The distribution of a common component is modified according to the macroeconomic conditions so that migrations towards riskier credit classes are more or less probable depending on whether the conditions are adverse or favorable. Probabilities of migration towards more secure credit ratings are adjusted in the opposite direction: they are higher (lower) under favorable (adverse) macroeconomic conditions. The macroeconomic drivers of the adjustment are represented by tendency variables. A tendency variable is assigned to every non-default credit class. In sum, the technique allows to model empirically observed variability of credit-rating migrations depending upon the phase of a business cycle.

Similar to the common factor models, see among others Bangia et al. (2002) or in Hull and White (2004), the common component renders individual migrations dependent. A distinctive feature is the path-dependence: its distribution depends upon the macroeconomic conditions.

In the existing coupling schemes, discussed in Kaniovski and Pflug (2007) and Wozabal and Hochreiter (2012), tendency variables do not evolve in time. In this paper, a time-homogeneous Markovian dynamics of tendency variables is considered. The simplest variant of such dynamics is implemented in the models employing a regime-switching matrix. They deal typically with a single tendency variable. See, among others, Bangia et al. (2002), Frydman and Schuermann (2008), or Fei et al. (2012). In particular, Fei et al. (2012) advocate a Mixture of Markov Chains (MMC) model as an efficient technique to account for stochastic business cycle effects. The use of the dynamic setting is twofold in our case. First, its solution serves as a benchmark to validate the results of the static setting. Second, its steady-state distribution is evaluated to check time-homogeneity of the probabilities assigned to the macroeconomic scenarios.

3 Macroeconomic scenarios and the respective conditional migration probabilities

Consider a portfolio where debtors are non-homogeneous in their credit ratings. Let there be $M \geq 1$ non-default credit classes. Numbering them in a descending order, let us assign 1 to the most secure assets, while the next to default credit class is indexed by M. Defaulted firms receive the index $M + 1$.

In the CreditMetrics approach, see Gupton et al. (1997), an $M \times (M + 1)$ Markovian transition matrix P is estimated. Its entry $P_{i,j}$ equals the probability that a debtor belonging to a credit class i at time t will move to a credit rating j at time $t+1$, $t = 1, 2, \ldots$. The $(M + 1)$-th row of P is never quoted. According to our notation, it corresponds to defaulted debtors. It is conventionally assumed that a defaulted debtor never returns to business, at least under its original name. Conceptually, there is a problem associated with withdrawn ratings. They are referred to as Not Rated (NR) and typically ignored or treated separately.

P is believed to reflect an average or a typical, rather than an instantaneous, market situation. This inability to account for the current macroeconomic conditions has been criticized in the literature on credit risk. See, among others, Altman (1998), Bangia et al.

(2002) and Korolkiewicz and Elliott (2008).

Denote by $\mathcal{N}$ the initial size of the portfolio. To trace individual migrations, assign a number $n = 1, 2, \ldots, \mathcal{N}$ to every debtor in the portfolio at time $t = 1$. Set $X_n(t)$ for the credit rating at time $t \geq 1$ of the debtor numbered by n. Since $X_n(t)$ is a time-homogeneous Markov chain, it will suffice to consider a transition from time $t = 1$ to time $t = 2$:

$$X_n(2) = \delta_n \xi_n + (1 - \delta_n)\eta_n. \tag{1}$$

Here, $\xi_n(\eta_n)$ is interpreted as an *idiosyncratic (common)* component in the migration from $X_n(1)$ to $X_n(2)$. The probability $\mathbb{E}\delta_n$ of success of the Bernoulli random variable δ_n determines the frequency of idiosyncratic migrations. The CreditMetrics' case, where all migrations are independent, corresponds to $\mathbb{P}\{\delta_n = 1\} = 1$. The families of random variables $\{\delta_n\}$, $\{\xi_n\}$ and $\{\eta_n\}$ are independent. The distribution of ξ_n is given by the $X_n(1)$-th row of P. Relation (1) can be interpreted as an agent-based model of dependent credit-rating migrations. A mechanism for adjusting the distribution of η_n according to macroeconomic scenarios is described next.

The macroeconomic factors affecting debtors belonging to the credit class i can be either *favorable* or *adverse*. They are encoded as 1 or 0. For M credit classes, a binary M-vector $\vec{\chi}$ describes a state of the economy. Its i-th coordinate χ_i encodes the macroeconomic factors affecting the credit class i. There are 2^M such vectors.

A typical model involving a regime-switching matrix assumes no differentiation among the macroeconomic factors affecting different credit classes. Consequently, there are two macroeconomic states and χ is a scalar. Since a contraction is an adverse macroeconomic scenario, the corresponding $\chi = 0$. An expansion is a favorable outcome, consequently, it is encoded as $\chi = 1$.

The conditional migration matrix is defined by the following relations:

$$P_{i,j}(1) = \begin{cases} P_{i,j}/P_i & \text{if } j \leq i, \\ 0 & \text{if } j > i; \end{cases} \quad \text{and} \quad P_{i,j}(0) = \begin{cases} P_{i,j}/(1 - P_i) & \text{if } j > i, \\ 0 & \text{if } j \leq i. \end{cases} \tag{2}$$

Here $P_i = P_{i,1} + P_{i,2} + \ldots + P_{i,i}$. These adjustments of the historical migration probabilities imply more (less) likely migrations towards riskier credit classes under adverse (favorable) macroeconomic conditions. Probabilities of migration towards more secure credit ratings are higher (lower) under favorable (adverse) macroeconomic conditions. In particular, $P_{i,j}/(1 - P_i) > P_{i,j}$ for $j > i$. This inequality holds true because $1 - P_i < 1$. Consequently, as compared to their historical counterparts, the conditional probabilities of migrating towards more risky credit classes increase $\frac{1}{1-P_i}$ times for debtors from the credit class i under adverse macroeconomic conditions.

The conditional distribution of η_n is defined in the following way:

$$\mathbb{P}\{\eta_n = j \mid \vec{\chi}\} = P_{X_n(1),j}(\chi_{X_n(1)}).$$

That is, the conditional probabilities $P_{i,j}(\cdot)$ and $\hat{P}_{i,j}(\cdot)$ depend only upon the i-th coordinate χ_i of $\vec{\chi}$.

According to formulas (1) and (2), a debtor from a credit class i migrates to a credit class j with probability $P_{i,j}$ or $P_{i,j}(1)$ ($P_{i,j}(0)$). Which of these possibilities takes place depends on the realization of the corresponding mixing variable δ_n. If it assumes the value 1, $P_{i,j}$ acts as the transition probability, while 0 implies $P_{i,j}(1)$ or $P_{i,j}(0)$ depending on whether the macroeconomic conditions for i are favorable or adverse. This individual randomizing behavior gives rise to the following conditional probabilities $\hat{P}_{i,j}(\cdot)$ governing migrations of the whole pool of debtors from credit the class i:

$$\hat{P}_{i,j}(1) = \begin{cases} q_i P_{i,j} + (1-q_i)\frac{P_{i,j}}{P_i} & \text{if } j \leq i, \\ q_i P_{i,j} & \text{if } j > i; \end{cases} \quad \hat{P}_{i,j}(0) = \begin{cases} q_i P_{i,j} & \text{if } j \leq i, \\ q_i P_{i,j} + (1-q_i)\frac{P_{i,j}}{1-P_i} & \text{if } j > i. \end{cases} \tag{3}$$

Here, q_i denotes $\mathbb{E}\delta_n$, common for all debtors n belonging to the credit class i. Since creditworthiness is the only classification criterion, debtors belonging to a credit class are not distinguishable. Consequently, their probabilities of success have to coincide.

Conditional migration probabilities $\hat{P}_{i,j}(\cdot)$ characterize the whole pool of debtors in the credit class i. In fact, the weights q_i and $1-q_i$ are frequencies of the two possible migration patterns, idiosyncratic or dependent, in the pool. Even if there can be no single debtor in credit class i whose migrations are governed by $\hat{P}_{i,j}(\cdot)$, these values can be interpreted as conditional migration probabilities of a *representative agent*. Probabilities $\hat{P}_{i,j}(\cdot)$ are *instantaneous*, because they depend upon a particular macroeconomic state. The values q_i are estimated using migration counts as inputs. For some portfolios and particular credit classes i, these parameters can be equal to 1. Then migrations in the credit class i are idiosyncratic, as it is postulated by the CreditMetrics framework. A value $q_i < 1$ indicates a dependence pattern between historical migrations. Formulas given next allow to quantitatively characterize its effect on the riskiness of the portfolio.

To quantify the impact of macroeconomic factors on migrations of the whole pool of debtors in the credit class i, let us compare probabilities $\hat{P}_{i,j}(\cdot)$ and their historical counterparts $P_{i,j}$. For this purpose, consider the percentage of variation $\Delta_{i,j}(\chi_i)$ of the migration probability $P_{i,j}$

$$\Delta_{i,j}(\chi_i) = \frac{\hat{P}_{i,j}(\chi_i) - P_{i,j}}{P_{i,j}} 100.$$

For $j \leq i$ ($j > i$) this is an upgrading (downgrading) probability. The corresponding formulas are summarized in Table 1. Let us interpret them conceptually.

Assume that $P_i > 1/2$. That is, the credit rating i less likely worsens than improves or remains unchanged. Such an assumption holds true for all historical migration matrices given in this paper. Then $(1-q_i)\frac{1-P_i}{P_i} < (1-q_i) < (1-q_i)\frac{P_i}{1-P_i}$ for every $q_i \in (0,1)$. Referring to Table 1, we conclude, first, that *macroeconomic factors have a weaker effect on an upgrading than on a downgrading probability* and, second, that *an upgrading probability under favorable macroeconomic conditions increases less than a downgrading probability under adverse macroeconomic conditions.* Characterizing the impact of macroeconomic factors on conditional migration probabilities, Fei et al. (2012) came to a similar conclusion: "The gap between the naïve and MMC estimates is visibly larger in contraction than in expansion." See p. 14. These authors call naïve the estimates based on historical migration probabilities.

According to Table 1, the percentage of decrease of a downgrading probability under favorable macroeconomic conditions coincides with the percentage of decrease of its upgrading counterpart under adverse macroeconomic conditions. Consequently, only one of them can be quoted in what follows next.

Table 1: Conditional vs. historical migration probabilities, percentage of variation.

	Upgrading	Downgrading
$\chi_i = 1$	$(1-q_i)\frac{1-P_i}{P_i}100$	$-(1-q_i)100$
$\chi_i = 0$	$-(1-q_i)100$	$(1-q_i)\frac{P_i}{1-P_i}100$

Involving all credit classes, a macroeconomic scenario corresponds to a binary M-vector $\vec{\chi}$. In order to distinguish between 2^M different macroeconomic scenarios, we can number them. Interpreting coordinates of $\vec{\chi}$ as a binary representation of an integer, let us number the vectors in a descending order of the corresponding integers so that $\vec{\chi}^{(1)} = (1, 1, \ldots, 1)$, while $\vec{\chi}^{(2^M)} = (0, 0, \ldots, 0)$. In particular, the four possible scenarios for $M = 2$ are listed in the following order:

$$\vec{\chi}^{(1)} = (1, 1), \quad \vec{\chi}^{(2)} = (1, 0), \quad \vec{\chi}^{(3)} = (0, 1), \quad \vec{\chi}^{(4)} = (0, 0).$$

To assess the likelihood of every scenario, we need a distribution over the set $\{0, 1\}^M$ of all of them. Assigning the probability π_j to the j-th binary M-vector $\vec{\chi}^{(j)}$, the distribution can be represented as a vector $\vec{\pi} = (\pi_1, \pi_2, \ldots, \pi_{2^M})$ as well. A random vector $\vec{\Pi}$ whose distribution is $\vec{\pi}$ is called a *tendency vector*. Its coordinates Π_i are referred to as *tendency variables*. Π_i represents the impact of macroeconomic factors on migrations in the credit class i. The probabilities π_j are estimated using migration counts as the input. If $\pi_j = 0$ for some j, the corresponding macroeconomic scenario is not feasible – it never comes true. The simplest conceptual interpretation is having π_1 (π_{2^M}), probability of the scenario favorable (adverse) for all debtors.

The choice of $\vec{\pi}$ is not arbitrary. Given formulas (2), the unconditional distribution of η_n coincides, by the formula of total probability, with the X_n-th row of P and, consequently, each individual migration in the credit class i is governed unconditionally by the mixture $q_i P_{i,j} + (1 - q_i)P_{i,j} = P_{i,j}$, if

$$\mathbb{P}\{\Pi_i = 1\} = P_i, \quad i = 1, 2, \ldots, M. \tag{4}$$

Consequently, to retain a fundamental assumption of the CreditMetrics approach regarding time-homogeneity of transition probabilities summarized in P, relations (3) have to hold true.

Fitting the model to historical credit rating migration data, $\vec{q} = (q_1, q_2, \ldots, q_M)$ and $\vec{\pi}$ have to be estimated, while a historical matrix P is assumed to be known. That is, a key element of the CreditMetrics approach – the transition matrix – is modified by means of $\vec{q}$ and $\vec{\pi}$ in order to match the phases of a business cycle. The parameters summarized in $\vec{q}$ account for the particular combination of assets in the portfolio and their exposure to the macroeconomic factors. The macroeconomic scenarios that can be observed for this portfolio during a business cycle correspond to the non-zero coordinates of $\vec{\pi}$ and their probabilities are given by these coordinates. Recall that all possible 2^M scenarios are numbered. That is, there is a one-to-one correspondence between them and the natural numbers 1, 2, ..., 2^M. In other words, the function of $\vec{\pi}$ is twofold: to indicate feasible macroeconomic scenarios and to characterize their likelihood. In the long run, over several business cycles, the driver responsible for the average dynamics – the historical matrix P – remains the same as in the CreditMetrics setting. A model for propagation of a business cycle is described next.

Assume that tendency vectors evolve as a time-homogeneous Markov chain, $\vec{\Pi}^t$, $t \geq 1$. Let $\mathcal{P} = (p_{i,j})$ be a Markovian matrix with 2^M rows and 2^M columns governing this dynamics. That is, the macroeconomic scenario encoded by a binary vector $\vec{\chi}^{(j)}$ follows the macroeconomic scenario corresponding to $\vec{\chi}^{(i)}$ with probability $p_{i,j}$ or

$$\mathbb{P}\{\vec{\Pi}^{t+1} = \vec{\chi}^{(j)} \mid \vec{\Pi}^t = \vec{\chi}^{(i)}\} = p_{i,j}.$$

Then the distribution of $\vec{\Pi}^t$, $t \geq 2$, given that the distribution of $\vec{\Pi}^1$ is $\vec{\pi}$, will be $\vec{\pi}\mathcal{P}^{t-1}$. Here, $\mathcal{P}^{t-1}$ stands for the $(t-1)$-th power of $\mathcal{P}$. In particular,

$$\mathbb{P}\{\vec{\Pi}^t = \vec{\chi}^{(j)}\} = [\vec{\pi}\mathcal{P}^{t-1}]_j,$$

where $[\vec{\pi}\mathcal{P}^{t-1}]_j$ denotes the j-th coordinate of the vector $\vec{\pi}\mathcal{P}^{t-1}$.

Observe that this setting generalizes, in a certain sense, the models employing regime-switching matrices. For example, Bangia et al. (2002), analyzing credit-rating migrations of American debtors, consider a (quarterly) 2×2 regime-switching matrix

$$\begin{pmatrix} \theta & 1-\theta \\ 1-\phi & \phi \end{pmatrix}.$$

Here, θ (ϕ) stands for the probability that a quarter of expansion (contraction) will be followed by a quarter of expansion (contraction). Correspondingly, $1-\theta$ is the probability that a quarter of contraction follows a quarter of expansion, and $1-\phi$ is the probability that a quarter of expansion follows a quarter of contraction.

With this observation in mind, an important feature of our approach becomes apparent. Our *economy is synthetic*: it comprises firms from different countries. What is common about them is that their creditworthiness is rated by the same agency. Consequently, it is neither a conventional nor an easy task to identify recessions and expansions in such an economy. In this sense, we argue about *non-observability* of tendency variables. A sub-economy comprising debtors having the same creditworthiness is a natural research object in this synthetic economy.

In particular, let $M = 2$. The corresponding levels of creditworthiness are represented by the investment-grade and the non-investment-grade debtors. Respectively, there are four states of the macro-economy: $(1, 1)$, $(1, 0)$, $(0, 1)$ and $(0, 0)$. Given a historical 2×3 migration matrix P, there are four conditional matrices corresponding to the above four macroeconomic states. They are formed by the respective $P_{i,j}(\cdot)$. In particular, with the outcome $(0, 1)$ the following matrix:

$$\begin{pmatrix} 0 & \frac{P_{1,2}}{1-P_1} & \frac{P_{1,3}}{1-P_1} \\ \frac{P_{2,1}}{P_2} & \frac{P_{2,2}}{P_2} & 0 \end{pmatrix}$$

is associated. Since in this case $\chi_1 = 0$ ($\chi_2 = 1$), its first (second) row is formed by $P_{1,j}(0)$ ($P_{2,j}(1)$). Consequently, migrations of every debtor in principle can be governed by five matrices: by the historical one and these four conditional matrices.

Not all of these potential possibilities need to occur. Consider, for example, the investment-grade debtors. If $q_1 = 1$, the credit rating of each of them evolves according to $P_{1,j}$. If $q_1 < 1$, the migration is governed by $P_{1,j}$ with probability q_1 and by $P_{1,j}(1)$ or $P_{1,j}(0)$ with probability $1 - q_1$.

Characterizing migrations of the entire pool of debtors, individual behaviors described by these conditional probabilities have to be taken into account according to their frequencies in the pool. Then the conditional probabilities $\hat{P}_{i,j}(\cdot)$ have to be considered. For example, the outcome $(1, 1)$ means that macroeconomic conditions are favorable for both credit classes. It occurs with probability $\mathbb{P}\{\{\Pi_1 = 1\} \cap \{\Pi_2 = 1\}\}$. Under this macroeconomic scenario, migrations of a representative debtor will be governed by

$$\begin{pmatrix} q_1 P_{1,1} + 1 - q_1 & q_1 P_{1,2} & q_1 P_{1,3} \\ q_2 P_{2,1} + (1 - q_2)\frac{P_{2,1}}{P_2} & q_2 P_{2,2} + (1 - q_2)\frac{P_{2,2}}{P_2} & q_2 P_{2,3} \end{pmatrix}.$$

The first (second) row of this conditional migration matrix corresponds to a representative investment-grade (non-investment grade) debtor. It is formed by $\hat{P}_{1,j}(1)$ ($\hat{P}_{2,j}(1)$). If $q_1 \neq 1$ and $q_2 \neq 1$, the probabilities of migration towards more secure (risky) credit classes are larger (smaller) here than their historical counterparts.

The matrix $\mathcal{P}$, an analog of the regime-switching matrix, is formed now by probabilities of the events that a macroeconomic state (i_1, i_2) will be followed by (j_1, j_2). This is a 4×4 matrix. According to the agreed way of numbering of macroeconomic scenarios, $p_{1,4}$ is the probability that $(1, 1)$ is followed by $(0, 0)$. That is, a macroeconomic state favorable for the whole synthetic economy is followed by an adverse for all debtors state. Within the approach employing a regime-switching matrix, Fei et al. (2012) introduced a regime-switching matrix with three regimes defined as "expansion, 'mild' recession and 'severe' recession, where mild and severe are qualified in terms of the time-length or severity measured, say as the percentage decrease in real GDP growth." See p. 8.

Fitting a dynamic model to historical credit rating migration data, additionally to $\vec{q}$ and $\vec{\pi}$, the Markovian matrix $\mathcal{P}$ has to be estimated.

4 Input data

For a description of the dataset see Appendix 1. The unknown parameters are estimated by the maximum likelihood method. The estimator is given in Appendix 2.

With a reduction (merging CCC, CC and C in a single credit class C) of the basic S&P's classification, $M = 7$ credit ratings are termed as AAA, AA, A, BBB, BB, B and

C. Because of the computational complexity of the corresponding estimators, the dynamic setting is tested for $M = 2$. In this case, investment-grade debtors are characterized by the S&P's ratings from AAA to BBB, while non-investment-grade debtors occupy the riskier ratings, starting from BB. Estimates for the static setting with $M = 2$ serve as a benchmark for interpreting the parameters obtained for $M = 7$.

The historical matrices P for $M = 7$ and $M = 2$ are estimated as the time averages:

$$\begin{pmatrix} 0.8948 & 0.0986 & 0.0047 & 0.0008 & 0 & 0 & 0 & 0.0011 \\ 0.0062 & 0.9012 & 0.0868 & 0.0045 & 0.0002 & 0.0007 & 0.0002 & 0.0002 \\ 0.0010 & 0.0356 & 0.9032 & 0.0562 & 0.0020 & 0.0006 & 0.0004 & 0.0010 \\ 0.0012 & 0.0047 & 0.0561 & 0.8825 & 0.0468 & 0.0063 & 0.0009 & 0.0015 \\ 0.0006 & 0.0033 & 0.0097 & 0.1102 & 0.7890 & 0.0747 & 0.0053 & 0.0072 \\ 0.0007 & 0.0012 & 0.0042 & 0.0107 & 0.0939 & 0.8051 & 0.0510 & 0.0332 \\ 0.0015 & 0 & 0.0015 & 0.0029 & 0.0205 & 0.1406 & 0.5717 & 0.2613 \end{pmatrix}, \tag{5}$$

$$\begin{pmatrix} 0.9779 & 0.0211 & 0.0010 \\ 0.0729 & 0.8957 & 0.0314 \end{pmatrix}.$$

Observe that P corresponding to $M = 2$ exhibits the monotonicity property, while its counterpart for $M = 7$ does not. A detailed discussion of this property and its implications is given on p. 457 of Bangia et al. (2002).

The respective probabilities P_i read:

$$(0.8948,\ 0.9073,\ 0.9398,\ 0.9445,\ 0.9127,\ 0.9158,\ 0.7387), \quad (0.9779,\ 0.9686). \tag{6}$$

Entries of the matrices

$$\begin{pmatrix} 1.0000/0 & 0/0.9370 & 0/0.0446 & 0/0.0079 & 0/0 & 0/0 & 0/0 & 0/0.0105 \\ 0.0069/0 & 0.9932/0 & 0/0.9372 & 0/0.0490 & 0/0.0023 & 0/0.0077 & 0/0.0023 & 0/0.0015 \\ 0.0011/0 & 0.0392/0 & 0.9611/0 & 0/09338 & 0/0.0328 & 0/0.0107 & 0/0.0060 & 0/0.0167 \\ 0.0014/0 & 0.0052/0 & 0.0596/0 & 0.9344/0 & 0/0.8437 & 0/0.1129 & 0/0.0165 & 0/0.0269 \\ 0.0006/0 & 0.0036/0 & 0.0104/0 & 0.1166/0 & 0.8644/0 & 0/0.8561 & 0/0.0607 & 0/0.0832 \\ 0.0008/0 & 0.0013/0 & 0.0044/0 & 0.0113/0 & 0.1029/0 & 0.8791/0 & 0/0.6064 & 0/0.3936 \\ 0.0016/0 & 0/0 & 0.0016/0 & 0.0031/0 & 0.0225/0 & 0.1535/0 & 0.7740/0 & 0/1.0000 \end{pmatrix}$$

and

$$\begin{pmatrix} 1.0000/0 & 0/0.9539 & 0/0.0461 \\ 0.0746/0 & 0.9254/0 & 0/1.0000 \end{pmatrix}$$

equal $P_{i,j}(1)/P_{i,j}(0)$.

5 Estimates and their interpretation

First, consider estimates for the setting without a macroeconomic dynamics. If $M = 2$, then

$$\vec{q} = (0.9822, 0.8788), \quad \pi_1 = 0.9496,\ \pi_2 = 0.0283,\ \pi_3 = 0.0190,\ \pi_4 = 0.0031.$$

For $M = 7$,

$$\vec{q} = (0.8373, 0.9078, 0.7991, 0.9060, 0.8396, 0.9008, 0.7728) \tag{7}$$

and the support of $\vec{\pi}$ consists of 11 binary vectors. The corresponding probabilities are:

$$\pi_1 = 0.5747,\ \pi_2 = 0.1137,\ \pi_8 = 0.0271,\ \pi_9 = 0.0555,\ \pi_{22} = 0.0500,\ \pi_{24} = 0.0102,$$

$$\pi_{33} = 0.0144,\ \pi_{34} = 0.0314,\ \pi_{35} = 0.0179,\ \pi_{65} = 0.0762,\ \pi_{100} = 0.0290.$$

Only the macroeconomic scenarios encoded by these binary vectors occur with a positive probability.

For example, for $M = 7$ consider the outcome numbered by 8. Its probability is $\pi_8 = 0.0271$. According to our numbering of binary strings, $\vec{\chi}^{(8)} = (1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0)$. In fact, the binary vector $(1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1)$ is numbered first and the corresponding sum equals: $1 \cdot 2^6 + 1 \cdot 2^5 + \ldots + 1 \cdot 2^0 = 128$. For the vector $(1,\ 1,\ 1,\ 1,\ 0,\ 0,\ 0)$ the sum is $1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 128 - 7$ and, consequently, its number is eight. Since 1 occupies the first four positions, this macroeconomic scenario is favorable for the debtors rated by AAA, AA, A and BBB, in other words, for the investment-grade debtors, while it is adverse for the debtors rated by BB, B and C, or the non-investment-grade ones, because 0 stands at the remaining three positions. Consequently, depending upon whether the synthetic economy in question is formed mainly by investment-grade or mainly by non-investment-grade debtors, expansion or contraction features will prevail. As a general term, a mild expansion or a mild recession could be appropriate in this case. Remark that an analog of this macroeconomic scenario for $M = 2$ is $(1, 0)$. Its probability is $\pi_2 = 0.0283$. The probabilities evaluated for $M = 7$ and $M = 2$ match each other pretty well: the relative error is approximately 4%. Returning to the case of $M = 7$, observe that under this macroeconomic scenario migrations of every debtor can be governed either by the historical matrix P or by the following conditional migration matrix:

$$\begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0069 & 0.9932 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0011 & 0.0392 & 0.9611 & 0 & 0 & 0 & 0 & 0 \\ 0.0014 & 0.0052 & 0.0596 & 0.9344 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8561 & 0.0607 & 0.0832 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6064 & 0.3936 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix}.$$

Its first four rows contain $P_{i,j}(1)$, $j = 1,\ 2, \ldots,\ 8$. The remaining three rows consist of $P_{i,j}(0)$, $j = 1,\ 2,\ \ldots,\ 8$. To characterize migrations of the entire pool of debtors under this macroeconomic scenario, the following conditional matrix has to be considered:

$$\begin{pmatrix} 0.9119 & 0.0826 & 0.0039 & 0.0007 & 0 & 0 & 0 & 0.0009 \\ 0.0063 & 0.9097 & 0.0788 & 0.0041 & 0.0002 & 0.0006 & 0.0002 & 0.0001 \\ 0.0010 & 0.0363 & 0.9148 & 0.0449 & 0.0016 & 0.0005 & 0.0003 & 0.0008 \\ 0.0012 & 0.0047 & 0.0564 & 0.8874 & 0.0424 & 0.0057 & 0.0008 & 0.0014 \\ 0.0005 & 0.0028 & 0.0081 & 0.0925 & 0.6624 & 0.2000 & 0.0142 & 0.0195 \\ 0.0006 & 0.0011 & 0.0038 & 0.0096 & 0.0846 & 0.7252 & 0.1061 & 0.0689 \\ 0.0012 & 0 & 0.0012 & 0.0022 & 0.0158 & 0.1087 & 0.4418 & 0.4291 \end{pmatrix}.$$

Its entries are the weighted sums of the probabilities governing individual migrations. For the credit class i, the weights are q_i and $1 - q_i$. That is, its first four rows are formed by $\hat{P}_{i,j}(1)$, $j = 1, 2, \ldots, 8$, while the remaining three contain $\hat{P}_{i,j}(0)$, $j = 1, 2, \ldots, 8$. Since $\pi_8 = 0.0271$, we conclude that, according to our model, this matrix governs 2.7% of all credit-rating migrations in the dataset.

Table 2: $M = 7$, conditional vs. historical migration probabilities, percentage of variation.

	Upgrading		Downgrading
Credit rating\Tendency	Favorable	Adverse	Favorable
AAA	1.91	-16.27	138.39
AA	0.94	-9.22	90.24
A	1.29	-20.09	313.63
BBB	0.55	-9.40	159.97
BB	1.53	-16.04	167.69
B	0.91	-9.92	107.90
C	8.04	-22.72	64.23

Inserting the probabilities P_i from (6) and the weights $\vec{q}$ given in (7) into the formulas presented in Table 1, the percentages of variation of upgrading and of downgrading probabilities against their historical counterparts are estimated for the case of $M = 7$. These values are summarized in Table 2. Instead of the numbers $1, 2, \ldots, 7$ corresponding to the index i in Table 1, the credit classes in Table 2 are referred to according to their conventional names, AAA, AA, $\ldots$, C. Also, the characterization of a macroeconomic outcome in terms of the respective tendency variable, $\chi_i = 1$ or $\chi_i = 0$, is substituted by the conceptual characterization, favorable or adverse. Note that, if a historical probability exceeds its conditional counterpart, the corresponding percentage is negative.

Using the 8-th column of the matrix P in (4), the probabilities P_i given in (5) and the weights $\vec{q}$ from (7) as inputs, the upper and lower bounds for the conditional default probabilities are evaluated according to the formulas (3). The values are quoted in Table 3. Both adverse and favorable for the respective credit classes macroeconomic scenarios are covered. For each credit class, the bounds are not symmetric with respect to the historical value: the lower bound is always tighter than the upper bound. The interval characterizes the magnitude of variation of the default likelihood. Such an effect is consistent with the original idea of coupling schemes as an instrument for modeling default cascades or, equivalently, heavy tails of the loss distribution that are typical for an economic downturn. See Kaniovski and Pflug (2007). To interpret conceptually this variation of the conditional default probabilities, their frequencies during the period of observation have to be taken into account. For example, consider the credit class BBB. The corresponding $i = 4$. Since $P_4 = 0.9445$, the tendency variable Π_4 assumes, by (3), the value $1(0)$ with probability $0.9445(0.0555)$. Then the default probability $0.0014\,(0.0039)$ characterizes $94.45\%\,(5.55\%)$ of defaults of the debtors rated at BBB. In other words, in 94.45% of cases the probability of default is 9.40% smaller than the historical value 0.0015, while in 5.55% of cases it is 159.97% larger than this value.

Table 3: Bounds of conditional default probabilities.

Tendency\Credit rating	AAA	AA	A	BBB	BB	B	C
Favorable	0.0009	0.0001	0.0008	0.0014	0.0061	0.0298	0.2019
Adverse	0.0026	0.0002	0.0042	0.0039	0.0195	0.0689	0.4291

The distribution of macroeconomic scenarios estimated for the setting without a macroeconomic dynamics accounts for the average macroeconomic condition during the period of observation. Inside this time interval, there are expansions (subperiods favorable for all debtors) as well as recessions (adverse for all debtors subperiods) and intermediate states, mild expansions or mild recessions, that are favorable for some debtors and adverse for the rest of the pool. Introducing a dynamics of macroeconomic states allows to identify the statistical regularities of the succession of these phases of a business cycle. The following $\vec{q}$, $\vec{\pi}$ and $\mathcal{P}$ were estimated for the dynamic setting with $M = 2$:

$$(0.9824, 0.8787), \quad (0.9486, 0.0293, 0.0200, 0.0021), \tag{8}$$

$$\begin{pmatrix} 0.9745 & 0.0153 & 0.0096 & 0.0056 \\ 0.5206 & 0.2721 & 0.1875 & 0.0198 \\ 0.4144 & 0.3126 & 0.2398 & 0.0332 \\ 0.3078 & 0.2746 & 0.2752 & 0.1424 \end{pmatrix}.$$

Observe that the estimates for $\vec{q}$ and $\vec{\pi}$ match their static counterparts very well: the error does not exceed 10^{-3}. Then the static model is sufficiently precise to meet the upper limit of 0.1% for the probability of undesirable credit events set by the Banking Supervision Accords (recommendations on banking regulations) – Basel I, Basel II and Basel III – issued by the Basel Committee on Banking Supervision (BCBS). In fact, the estimates according to a more sophisticated method differ at most by 10^{-3} from the simpler ones. Consequently, the error of the simpler estimator does not exceed 10^{-3}.

The Markov chain governing macroeconomic states is hidden because tendency variables are not observed: economic statistics does not report separately recessions (expansions) of firms rated as the investment-grade and as the non-investment-grade debtors. According to $\mathcal{P}$, the macroeconomic state $(1, 1)$, favorable for all debtors, deteriorates gradually: $p_{1,2}$ is 60% larger than $p_{1,3}$ and 173% larger than $p_{1,4}$. This is a plausible dynamics. A complete recovery, corresponding to $(1, 1)$, is the most likely next macroeconomic state for $(1, 0)$, $(0, 1)$ and $(0, 0)$. In fact, $p_{i,1}$, $i = 2, 3, 4$, are the largest entries in the respective rows of $\mathcal{P}$. This is again a natural dynamics given that $(1, 1)$ is the most likely macroeconomic scenario.

Dealing with a time-homogeneous Markov chain, an important question is whether it is stationary or not. In formal terms, it corresponds to the following relation between the transition matrix $\mathcal{P}$ of the chain and a distribution $\vec{\pi}$ over its states: $\vec{\pi}\mathcal{P} = \vec{\pi}$. A distribution satisfying this equation is called the *steady-state distribution* of the chain. Conceptually, this is a distribution that persists in time: on the one hand, starting from it, the chain will maintain this distribution at any time instant and, on the other hand, the distribution of the chain at t will converge as $t \to \infty$ to the steady-state for any choice of the initial distribution. For the given above Markovian matrix $\mathcal{P}$ the steady state distribution equals:

$$(0.9448,\ 0.0291,\ 0.0198,\ 0.0062).$$

Since these values deviate at most by 0.005 from the probabilities $\vec{\pi}$ quoted in (8), the hidden Markov chain seems to be stationary. Non-stationarity of this chain would imply varying in time probabilities of the macroeconomic scenarios.

6 Conclusions

Within the CreditMetrics approach, the conditional migration probabilities dependent upon the macroeconomic scenarios are proposed. Two cases are considered: with 2 and 7 non-default credit classes. The gap between conditional and the historical migration probabilities is estimated. It typically increases as the creditworthiness decreases. This is consistent with the commonly accepted view that riskier credit classes exhibit higher volatility and they are stronger affected by the macroeconomic conditions than more secure ones. In many cases, the variation of default probabilities exceeds 100%. The maximum percentage for the period from 1991 to 2013 is 313.63. This is a compelling argument in favor of the risk models based on the conditional probabilities, in particular, for stress testing. Upgrading probabilities do not exhibit the same magnitude of variation for every credit class: the percentage of increase of an upgrading probability due to favorable macroeconomic conditions is smaller than the percentage of increase of a downgrading probability due to averse macroeconomic conditions. This is consistent with the results reported in the literature. See Fei et al. (2012), for example.

Probabilities of feasible macroeconomic scenarios are evaluated. To test robustness of the estimates, a generalization of the model with macroeconomic scenarios evolving as a finite Markov chain is considered. For the case of two non-default credit classes, the estimates corresponding to the setting without and with the dynamics of macroeconomic states coincide. Since the dynamics of macroeconomic states is stationary, probabilities of the macroeconomic scenarios are time-homogeneous.

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Appendix 1

The credit ratings data for corporate and financial industry borrowers comes from S&Ps. We have used Ratingsdirect database and RatingsXpress dataset in DataStream to download the rating histories of all the available debtors from 35 OECD countries that had been rated by the S&Ps agency in the period from 1991 to the end of 2013. The rating histories were used to reconstruct the credit ratings as at the end of each calendar year. After removing empty or no-rating (NR) values and controlling for duplicates, we were left with 13304 debtors coming from 34 countries (Latvia had no borrowers identified). The following table lists the final sample composition:

Table 4: OECD countries covered.

Country	Corporate debtors	Financial debtors	Total
Australia	193	168	361
Austria	16	51	67
Belgium	22	39	61
Canada	441	157	598
Chile	23	10	33
Czech Republic	6	13	19
Denmark	19	31	50
Estonia	1	0	1
Finland	19	28	47
France	148	285	433
Germany	128	750	878
Greece	12	14	26
Hungary	10	18	28
Iceland	1	2	3
Ireland	30	71	101
Israel	4	3	7
Italy	52	153	205
Japan	413	211	624
Korea, Republic of	50	29	79
Luxembourg	30	56	86
Mexico	88	47	135
Netherlands	107	111	218
New Zealand	32	65	97
Norway	19	25	44
Poland	11	19	30
Portugal	12	32	44
Slovakia	3	6	9
Slovenia	0	1	1
Spain	26	69	95
Sweden	60	67	127
Switzerland	35	78	113
Turkey	10	21	31
United Kingdom	353	355	708
United States of America	4340	3605	7945
Total	6714	6590	13304

Appendix 2

The estimates are based on 23 years of observation: $t = 1$ ($T = 23$) corresponds to 1991 (2013). The coupling scheme considered in Kaniovski and Pflug (2007) was used. Since the static setting is a particular case, we present only the maximum likelihood function of the dynamic setting:

$$L(\vec{\pi},\vec{q},\mathcal{P}) = \prod_{t=1}^{T}\sum_{i=1}^{2^M}[\vec{\pi}\mathcal{P}^{t-1}]_i \prod_{m_1=1}^{M} v(t,\vec{\chi}^{(i)},m_1,\vec{q}).$$

Here,

$$v(t,\vec{\chi},m_1,\vec{q}) = \sum_{m_2=1}^{M+1} P_{m_1,m_2}(\chi_{m_1})(q_{m_1}+\frac{1-q_{m_1}}{P_{m_1,m_2}})^{I^t(m_1,m_2)} \prod_{j=1,\, j\neq m_2}^{M+1} q_{m_1}^{I^t(m_1,j)}$$

and $I^t(l,k)$ denotes the number of debtors who moved from the credit class l to the credit class k in the period t.

The linear equality constraint

$$\sum_{j=1}^{2^M}\pi_j = 1,$$

states that the values π_j form a probability distribution. To guarantee that at every time instant the i-th coordinate of a tendency vector assumes the value 1 with probability P_i, it is required that

$$\sum_{j=1}^{2^M}[\vec{\pi}\mathcal{P}^{t-1}]_j \mathbb{I}_{\{\chi_i^{(j)}=1\}} = P_i, \quad i=1,2,\ldots,M, \quad t=1,2,\ldots,T.$$

Here, $\mathbb{I}_{\{A\}}$ denotes the indicator function of a statement A. Since $\mathcal{P}$ is a Markovian transition matrix, the following equality constraints must hold true:

$$\sum_{j=1}^{2^M} p_{i,j} = 1, \quad i=1,2,\ldots,2^M. \tag{9}$$

To prevent transitions to the states $\vec{\chi}^{(i)}$ such that $\pi_i = 0$, it is required that:

$$[\vec{\pi}\mathcal{P}^{t-1}]_i \leq D\pi_i, \quad i=1,2,\ldots,2^M, \quad t=2,3,\ldots,T. \tag{10}$$

Here, D denotes a positive constant. (In case quoted here, $D = 150$ is used for the calculations.) Elements of $\vec{q}$, $\mathcal{P}$ and $\vec{\pi}$ belong to $[0,1]$.

In the static setting, we can get rid of constraints (9) and (10) by letting $\mathcal{P}$ be equal to the $2^M \times 2^M$ identity matrix.