Bounds on the Competence of a Homogeneous Jury

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Abstract

In a homogeneous jury the votes are exchangeable correlated Bernoulli random variables. We derive the bounds on a homogeneous jury's competence as the minimum and maximum probability of the jury being correct, which arise due to unknown correlations among the votes. The lower bound delineates the downside risk associated with entrusting decisions to the jury. In large and not-too-competent juries the lower bound may fall below the success probability of a fair coin flip - one half, while the upper bound may not reach a certainty. We also derive the bounds on the voting power of an individual juror as the minimum and maximum probability of her casting a decisive vote. The maximum is less than one, while the minimum of zero can be attained for infinitely many combinations of distribution moments.

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1 Introduction

The literature on Condorcet's Jury Theorem studies the expertise of a group of experts. In a criminal jury the experts are sworn jurors whose common purpose is to convict the guilty and acquit the innocent. The jury reaches its verdict by a formal vote. Under simple majority voting rule, half the total number of votes plus one vote will suffice to reach a verdict.[1]

As each juror faces a dichotomous choice, we model the juror's vote as a Bernoulli random variable. We can thus measure individual competence by a juror's probability of voting for the correct alternative, and collective competence by the probability that a simple majority of the jurors will vote correctly. A juror is said to be competent if his vote is more likely to be correct than incorrect.

Condorcet's Jury Theorem asserts that a group of competent and independent jurors is more likely than any single juror to select the correct alternative. This likelihood, called Condorcet's probability, tends to a certainty as jury size increases. The classic version of the theorem assumes independent votes (Young 1988, Boland 1989). However, the independence assumption is refuted by empirical evidence (Heard and Swartz 1998, Kaniovski and Leech 2009). Moreover, stochastic independence cannot be reconciled with commonalities and differences in jurors' preferences, information asymmetries and strategic behavior, as these factors will induce correlations between the votes. The more recent literature studies the conditions for which the theorem remains valid, also in the case of correlated votes.[2]

[1] Take the highest judicial authority of the United States, the U.S. Supreme Court. The court operates simple majority rule. A full bench comprises nine justices. Provided at least six justices are present, the votes of half the number of justices plus one will suffice to reach a verdict.

[2] See, Berg (1985, 1993), Ladha (1992, 1993, 1995), Berend and Sapir (2007), Peleg and Zamir (2008), Kaniovski (2009), Kaniovski and Zaigraev (2009). Like the original Condorcet's Jury Theorem, these generalizations assume sincere voting. Jury models that consider the incentive to acquire information show that sincere voting is irrational in the presence of informational asymmetries among jurors (e.g., Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1996, 1997)). In our analysis we abstract from the effect of the incentive to acquire information on collective competence. In this our approach follows the tradition of the classic Condorcet's Jury Theorem.

In this paper we obtain the minimum and maximum Condorcet probability for a given individual competence. The problem arises when individual competence is known, e.g. from past decisions, whereas correlations among the votes are not known, e.g. because the jurors have never voted together in the past. Our results thus contribute to a theory of optimal jury design when the recipient of jury expertise is risk averse. The lower bound, which is consistent with given individual competence, delineates the downside risk associated with entrusting decisions to the jury. This probability can be substantially lower than Condorcet's probability under the assumption of zero higher order correlations obtained for simple majority rule in Kaniovski (2009) and for an arbitrary voting rule in Kaniovski and Zaigraev (2009). The upper bound, which represents the best case scenario, can be lower than one.

Condorcet's Jury Theorem assumes a homogeneous jury in which all correlation coefficients are equal to zero, while we consider homogeneous juries with correlated votes. In either case the votes are exchangeable random variables. The probability of a voting profile depends on the total number of correct votes in it, but not on their order. For example, in a jury consisting of three jurors, the profiles $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ would be equally probable, as each contains two correct votes (denoted by 1). The homogeneous jury model is an example of a representative agent model common to social sciences.

The probabilities of voting profiles and Condorcet's probability are linear in the correlation coefficients. The non-negativity of the former probabilities thus imposes linear constraints in the optimization problem in which the latter probability is minimized or maximized. Finding correlation coefficients that minimize or maximize Condorcet's probability therefore amounts to solving a linear programming problem. Sections 2 and 3 describe the model and present the results. The linear programming approach used for evaluating the bounds on the competence of a homogeneous jury can also be used to evaluate the bounds on the voting power of an individual juror as the minimum and maximum probability of casting a decisive vote. Section 4 is dedicated to this problem. The last section provides a summary and discussion of the results. All proofs have been relegated to Appendix A. Appendices B and C provide two fully worked examples for small juries.

2 The model

We model juror i's vote as a realization v_i of a Bernoulli random variable V_i. Juror i is correct if $v_i = 1$, and incorrect if $v_i = 0$. Juror i's individual competence is measured by his marginal probability of being correct p_i, such that $P(V_i = 1) = p_i$ and $P(V_i = 0) = 1 - p_i$.

In a jury of n ($n \geq 3$) jurors, the n-tuple of votes $\mathbf{v} = (v_1, \ldots, v_n)$ is called a voting profile. There will be 2^n such voting profiles. Computing the probability of a correct verdict requires a probability distribution on the set of voting profiles, which is the joint probability distribution of n Bernoulli random variables. For independent Bernoulli random variables this distribution is given by

$$\bar{\pi}_{\mathbf{v}} = \prod_{i=1}^{n} p_i^{v_i} (1 - p_i)^{1-v_i}.$$

Bahadur (1961) obtained the joint probability distribution of n correlated Bernoulli random variables. Let $Z_i = (V_i - p_i)/\sqrt{p_i(1 - p_i)}$ for all $i = 1, 2, \ldots, n$, and

$$c_{i,j} = E(Z_i Z_j) \quad \text{for all} \quad 1 \leq i < j \leq n;$$

$$c_{i,j,k} = E(Z_i Z_j Z_k) \quad \text{for all} \quad 1 \leq i < j < k \leq n;$$

$$\ldots$$

$$c_{1,2,\ldots,n} = E(Z_1 Z_2 \ldots Z_n).$$

Here, $c_{i,j}$ is the Pearson product-moment correlation coefficient. Higher order coefficients measure dependence between the general tuples of votes. Let C_n^x denote the binomial coefficient $C_n^x = n!/[x!(n-x)!]$ for $n, x \in \mathbb{N}$, where $C_n^x = 0$ for $n < x$. There will be $\sum_{i=2}^n C_n^i = 2^n - n - 1$ correlation coefficients of all orders, which together with n marginal probabilities uniquely define the joint probability distribution of n correlated Bernoulli random variables:

$$\pi_{\mathbf{v}} = \bar{\pi}_{\mathbf{v}} \left(1 + \sum_{i<j} c_{i,j} z_i z_j + \sum_{i<j<k} c_{i,j,k} z_i z_j z_k + \cdots + c_{1,2,\ldots,n} z_1 z_2 \ldots z_n \right), \quad \text{where} \quad \sum_{\mathbf{v}} \pi_{\mathbf{v}} = 1. \quad (1)$$

In the above equation, $z_i = (v_i - p_i)/\sqrt{p_i(1-p_i)}$ denotes a realization of the random variable Z_i. In the following we assume that n and p are known but the correlation coefficients are not.

Bahadur's (1961) representation makes clear that first and second moments alone (expected values and a correlation matrix between the votes) do not uniquely define the joint probability distribution of correlated Bernoulli random variables, and hence also not Condorcet's probability. A complete specification of the probability distribution requires the inclusion of higher order coefficients, which are prohibitively numerous. This fact is important since the majority of existing extensions of Condorcet Jury Theorem to correlated votes discuss second-order correlations only (e.g., Boland 1989, Boland, Proschan and Tong 1989, Ladha 1992, Ladha 1993).

2.1 Homogeneous jury

This paper studies the simplest extension of Condorcet's Jury Theorem to correlated votes. We relax the independence assumption while maintaining the homogeneity of the jury, which is now is defined as:

Definition 1. *In a homogeneous jury each vote has an equal probability of being correct, and all correlation coefficients of the same order are equal.*

In a homogeneous jury the votes are exchangeable random variables. Formally, the joint probability distribution of an exchangeable sequence is invariant under permutation of its arguments. We can thus index the probabilities of voting profiles according to the number of incorrect votes (zeros) they contain. Intuitively, exchangeability introduces symmetry on the jurors.

Let $p_i = p \in (0.5, 1)$ for all $i = 1, 2, \ldots, n$ and

$$c_{i,j} = x_2 \quad \text{for all} \quad 1 \leq i < j \leq n;$$

$$c_{i,j,k} = x_3 \quad \text{for all} \quad 1 \leq i < j < k \leq n;$$

$$\ldots$$

$$c_{1,2,\ldots,n} = x_n.$$

The total number of correlation coefficients is $n-1$. Since the total number of incorrect votes $k(\mathbf{v}) = n - \sum_{i=1}^{n} v_i$ ranges from 0 to n, there will be $n+1$ distinct probabilities π_k, $k = 0, 1, \ldots, n$. From Bahadur's distribution (1), we obtain

$$\pi_k = p^{n-k}(1-p)^k + x_2 p^{n-k-1}(1-p)^{k-1}\left[C_{n-k}^2(1-p)^2 - C_{n-k}^1 C_k^1 p(1-p) + C_k^2 p^2\right] +$$
$$\ldots + x_j p^{n-k-\frac{j}{2}}(1-p)^{k-\frac{j}{2}}\left[C_{n-k}^j(1-p)^j - C_{n-k}^{j-1} C_k^1 p(1-p)^{j-1} + \ldots + (-1)^j C_k^j p^j\right] +$$
$$\ldots + (-1)^k x_n p^{\frac{n}{2}}(1-p)^{\frac{n}{2}}.$$

George and Bowman (1995) provide a more compact representation of the above probabilities. Let $\lambda_k = P(V_1 = 1, V_2 = 1, \ldots, V_k = 1)$, $k = 1, 2, \ldots, n$ (of course, $\lambda_1 = p$), and $\lambda_0 = 1$. Then,

$$\pi_k = \sum_{j=0}^{k}(-1)^j C_k^j \lambda_{n-k+j} = \Delta^k(\lambda_{n-k}), \qquad (2)$$

where $\Delta^k(\lambda_{n-k})$ denotes the k-th finite difference of λ_{n-k}. In the same paper, authors state the relation between $\{x_k\}$ and $\{\lambda_k\}$:

$$x_k = \frac{\sum_{j=0}^{k-2}(-1)^j C_k^j p^j \lambda_{k-j} + (-1)^{k-1}(k-1)p^k}{p^{\frac{k}{2}}(1-p)^{\frac{k}{2}}}, \qquad k \geq 2. \tag{3}$$

3 Competence of a jury

To avoid the need for a tie-breaking rule, we assume that n is *odd*. Condorcet's probability is the probability of at most $\frac{n-1}{2}$ failures in n Bernoulli trials:

$$M_{n,p}(x_2, x_3, \ldots, x_n) = M_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = \sum_{k=0}^{\frac{n-1}{2}} C_n^k \pi_k. \tag{4}$$

To illustrate the quantities involved, consider a numerical example for $n = 3$ and $p = 0.75$. Eight conceivable voting profiles may occur with the probabilities listed in Table 1.

Table 1: Examples of joint probability distributions ($n = 3$, $p = 0.75$)

v_1	v_2	v_3	k	π_k		
1	1	1	0	0.625	0.422	0.250
1	1	0	1	0	0.141	0.250
1	0	1	1	0	0.141	0.250
0	1	1	1	0	0.141	0.250
1	0	0	2	0.125	0.047	0
0	1	0	2	0.125	0.047	0
0	0	1	2	0.125	0.047	0
0	0	0	3	0	0.016	0
x_2				0.333	0	-0.333
x_3				0.770	0	-0.385
$M_{3,p}$				0.625	0.844	1.000

The competence of a jury is measured by Condorcet's probability $M_{3,p} = \pi_0 + 3\pi_1$. In the example on the left this probability is at its minimum of 0.625. In the case of independent votes this probability equals 0.844. On the right we have the maximum probability of one.

Notice distinct vanishing probabilities in the distributions corresponding to the bounds: (π_1, π_3) for the lower, and (π_2, π_3) for the upper. We shall see that the pattern behind the vanishing probabilities is the key to finding the correlation coefficients.

3.1 Linear programming problems

We seek the minimum and maximum Condorcet probabilities for given *odd* n and p:

$$\min_{x_2, x_3, \dots, x_n} M_{n,p}(x_2, x_3, \dots, x_n), \qquad \max_{x_2, x_3, \dots, x_n} M_{n,p}(x_2, x_3, \dots, x_n).$$

Since the correlation coefficients enter $M_{n,p}$ linearly, and the non-negativity of the probabilities π_k imposes $n+1$ linear constraints, finding correlation coefficients that minimize or maximize $M_{n,p}$ for given n and p amounts to solving the linear programming problems:

$$M_{n,p}(x_2, x_3, \dots, x_n) \to \min \quad \text{subject to} \quad \pi_k \geq 0, \quad k = 0, 1, \dots, n; \tag{5}$$

$$M_{n,p}(x_2, x_3, \dots, x_n) \to \max \quad \text{subject to} \quad \pi_k \geq 0, \quad k = 0, 1, \dots, n. \tag{6}$$

In view of George and Bowman's (1995) results one can substitute the correlation coefficients $\{x_k\}$ by probabilities $\{\lambda_k\}$.

The probabilities π_k depend on the parameters n, p and the unknowns $x_2, x_3, \dots, x_n$ according to (2). The above problems can be written in the standard form as:

$$\mathbf{a}^T\mathbf{x} \to \min \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}; \tag{7}$$

$$\mathbf{a}^T\mathbf{x} \to \max \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{8}$$

where $\mathbf{a} \in \mathbb{R}^{n-1}$, $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\mathbf{A} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$; and in the dual form as:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = -\mathbf{a}, \quad u_i \geq 0, \quad i = 0, 1, \ldots, n; \tag{9}$$

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = \mathbf{a}, \quad u_i \geq 0, \quad i = 0, 1, \ldots, n. \tag{10}$$

The Duality Theorem says that if $\mathbf{x}^*$ and $\mathbf{u}^*$ are the solutions of the primal Problem (7) and the corresponding dual Problem (9), then $\mathbf{a}^T\mathbf{x}^* = -\mathbf{b}^T\mathbf{u}^*$; while if $\mathbf{x}^*$ and $\mathbf{u}^*$ are the solutions of the primal Problem (8) and the corresponding dual Problem (10), then $\mathbf{a}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{u}^*$.[3] The above primal linear programming problems involve $n - 1$ variables and $n + 1$ constraints. Typically, problems of this size can only be solved numerically, e.g. using the simplex method. Fortunately, a closer look at the dual problems revealed that the solutions of the primal problems satisfy the systems of linear equations formed by vanishing probabilities π_k.

Theorem 1. *The solutions to Problem (5) satisfy the system of $n - 1$ linear equations:*

$$\pi_1 = 0, \ldots, \pi_{\frac{n-1}{2}} = 0, \pi_{\frac{n+3}{2}} = 0, \ldots, \pi_n = 0. \tag{11}$$

Moreover,

$$\min_{x_2, x_3, \ldots, x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = \frac{n(2p-1)+1}{n+1}.$$

The solutions to Problem (6) for $p < \frac{n+1}{2n}$ satisfy the system of $n - 1$ linear equations:

$$\pi_0 = 0, \ldots, \pi_{\frac{n-3}{2}} = 0, \pi_{\frac{n+1}{2}} = 0, \ldots, \pi_{n-1} = 0, \tag{12}$$

[3] For an exposition of duality theory, see, e.g., Korte and Vygen (2008, Ch. 3).

while for $p > \frac{n+1}{2n}$ they satisfy the system of $\frac{n+1}{2}$ linear equations:

$$\pi_{\frac{n+1}{2}} = 0, \ldots, \pi_n = 0. \tag{13}$$

For $p = \frac{n+1}{2n}$ the solutions to Problem (6) satisfy:

$$\pi_0 = 0, \ldots, \pi_{\frac{n-3}{2}} = 0, \pi_{\frac{n-1}{2}} = 1/C_n^{\frac{n-1}{2}}, \pi_{\frac{n+1}{2}} = 0, \ldots, \pi_n = 0.$$

Moreover,

$$\max_{x_2, x_3, \ldots, x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = \min\left\{\frac{2np}{n+1}, 1\right\}.$$

Proof of Theorem 1 is given in Appendix A. In Appendix B we provide a complete analysis for $n = 3$.

Remark (Uniqueness): Systems of linear equations (11) and (12) have unique solutions and the solutions of corresponding linear programming problems are therefore also unique. System (13) has a unique solution for $n = 3$, and infinitely many solutions for any odd $n \geq 5$.

By solving systems (11) and (12) one can find those values of correlation coefficients which minimize or maximize Condorcet's probability. In particular, minimizing Condorcet's probability, we get the following interesting result (see proof in Appendix A):

Corollary 1. *For any given odd n and p, Condorcet's probability attains a unique minimum for $x_2 = \frac{2p-1}{2p}$ or $\lambda_2 = \frac{3p-1}{2}$.*

Corollary 1 establishes that in a homogeneous jury, regardless of jury size n, Condorcet's probability is minimal when the second-order correlation coefficient $x_2 = \frac{2p-1}{2p}$. Figure 1 illustrates how $\min M_{n,p}, \max M_{n,p}$ vary in n and p. We have,

1. $\min M_{n,p} \to (2p-1) > 0$ for $n \to \infty$ and $p \in (0.5, 1)$;

2. $\min M_{n,p}$ is linear and increasing in p. For $p > \frac{3n-1}{4n}$, $\min M_{n,p} > 0.5$.

In the case of the upper bound, we have $\max M_{n,p} = 1$ for $p \geq \frac{n+1}{2n}$, while for $p < \frac{n+1}{2n}$:

1. $\max M_{n,p} \to 1$ for $n \to \infty$, $p \in (0.5, 1)$, and $\max M_{n,p} \geq 0.75$;

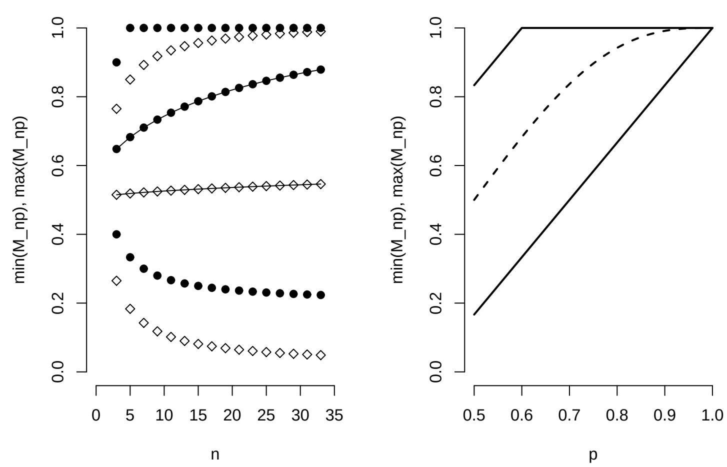
2. $\max M_{n,p}$ is linear and increasing in p.

Finally, the range of Condorcet's probabilities is given by

$$\mathcal{R} = \max M_{n,p} - \min M_{n,p} = \min \left\{ \frac{n-1}{n+1}, \frac{2n(1-p)}{n+1} \right\}.$$

We have $\mathcal{R} \to 0$ as $p \to 1$. In the case of a low p, $\mathcal{R} \in [0.5, 1)$ does not depend on p, and $\mathcal{R} \to 1$ as $n \to \infty$.

Figure 1: $\min M_{n,p}$ and $\max M_{n,p}$ as functions of n and p:



The left panel plots the bounds as a function of n for $p = 0.51$ (diamonds) and $p = 0.60$ (bullets). For a comparison we add Condorcet's probabilities for independent votes (overlaid lines). The right panel plots the minimum and the maximum (solid lines) as a function of p for $n = 5$, and Condorcet's probability for independent votes (dashed line).

4 Voting power of a juror

The classic measures of voting power by Penrose (1946) and Banzhaf (1965), or Shapley and Shubik (1954) have been criticized for treating all voting profiles as equally likely.[4] In the case of the Penrose-Banzhaf measure this assumption is equivalent to a binomial model, in which each vote has an equal probability of being for or against and all votes are stochastically independent. Both assumptions are widely refuted by empirical evidence (Gelman, Katz and Bafumi 2004).

The binomial model has been deemed appropriate in the absence of information on the future behavior of voters and the issues they will face. Such is the case at the constitutional stage of a voting assembly, in which voting weights and a decision rule are agreed. This information is sufficient in order to know who will be decisive in each voting profile. What is unknown at this stage is the probability with which each profile will occur. This probability will depend on the voting behavior. The uncertainty shrouding the latter gives rise to the bounds on voting power as the probability of casting a decisive vote.

Exchangeable votes offer the least restrictive stochastic model of individual voting behavior that still provides a degree of 'a prioriness' in the calculus of power. Exchangeability entails a stochastic representative agent model. It should be understood as an extension of the behavioral part of the calculus of power. The fact that, in general weighted voting games, the votes may carry different voting weights makes them obviously non-exchangeable in the common sense of the word. Our proposal is a compromise between a measure of voting power based on completely general probabilities in Laruelle and Valenciano (2004, 2005) and Kaniovski (2008), and the *a priori* view on the measurement of power advocated in Felsenthal and Machover (2005).

In a symmetric simple-majority game with an odd number of voters a voter is decisive if she

[4]For a history of ideas behind the measurement of voting power, see Felsenthal and Machover (2004). For a comprehensive treatment of the topic, see Felsenthal and Machover (1998).

breaks a tie. We shall therefore consider situations in which exactly $\frac{n}{2}$ votes are in favor so that the $n+1$'s vote is decisive, where n is taken to be *even*. There will be $C_n^{\frac{n}{2}}$ such profiles. In other words, we study the joint probability distribution of all votes except the vote whose power is being measured. The voting power is measured by the following probability:

$$P_{n,p}(x_2, x_3, \ldots, x_n) = C_n^{\frac{n}{2}} \pi_{\frac{n}{2}}.$$

Clearly, finding $\min P_{n,p}$ and $\max P_{n,p}$ amounts to finding $\min \pi_{\frac{n}{2}}$ and $\max \pi_{\frac{n}{2}}$. We can now formulate the corresponding linear programming problems.

4.1 Linear programming problems

We seek the minimum and maximum probabilities of a tie, $\min \pi_{\frac{n}{2}}$ and $\max \pi_{\frac{n}{2}}$, for given *even* $n \geq 2$ and $p \in (0.5, 1)$:

$$\min_{x_2, x_3, \ldots, x_n} \pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n), \qquad \max_{x_2, x_3, \ldots, x_n} \pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n).$$

Finding correlation coefficients that minimize or maximize $\pi_{\frac{n}{2}}$ for given n and p leads to the linear programming problems:

$$\pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n) \to \min \quad \text{subject to} \quad \pi_k \geq 0, \quad k = 0, 1, \ldots, n; \tag{14}$$

$$\pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n) \to \max \quad \text{subject to} \quad \pi_k \geq 0, \quad k = 0, 1, \ldots, n. \tag{15}$$

Although voting power is bounded between 0 and 1 as a probability, exchangeability imposes constraints on the joint probability distribution, so that these bounds may not be attained. However, it is easy to see that $\min P_{n,p} = 0$ indeed, which holds when $\pi_{\frac{n}{2}} = 0$. What is remarkable about this otherwise obvious result is that it holds for a infinite number of combinations $x_2, x_3, \ldots, x_n$, thereby the ubiquitous Pearson measure of dependence x_2 can be close to zero. This result is simple to prove so we omit its proof, except for $n = 4$ in Appendix C.[5] Instead we show that for $p > 0.5$, $\max P_{n,p} < 1$. Indeed, the equality $\max P_{n,p} = 1$ is attained only if $\pi_{\frac{n}{2}} = 1/C_n^{\frac{n}{2}}$, in which case all other profiles occur with zero probability. Our second theorem shows this to be impossible in a homogeneous jury.

Theorem 2. *The solutions to Problem (15) satisfy the system of $n-1$ linear equations:*

$$\pi_1 = 0, \ldots, \pi_{\frac{n}{2}-1} = 0, \pi_{\frac{n}{2}+1} = 0, \ldots, \pi_n = 0. \tag{16}$$

Thus,

$$\max_{x_2,x_3,\ldots,x_n} \pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n) = \frac{1-p}{C_{n-1}^{\frac{n}{2}}}, \quad x_2^* = 1 - \frac{n}{2p(n-1)}, \quad \pi_0 = 2p - 1,$$

and

$$\max_{x_2,x_3,\ldots,x_n} P_{n,p}(x_2, x_3, \ldots, x_n) = 2(1-p).$$

5 Summary

In a homogeneous jury votes are exchangeable random variable may be positively or negatively correlated. We evaluate the bounds on the probability of a homogeneous jury collectively reaching a correct decision when the exact dependences among the jurors are unknown. Typically, both the upper bound and the lower bound will depend on jury size n (n is odd) and the

[5]A proof in the general case is available from the authors upon request.

individual probability of casting a correct vote p. These are given by:

$$\min_{x_2,x_3,\ldots,x_n} M_{n,p}(x_2,x_3,\ldots,x_n) = \frac{n(2p-1)+1}{n+1};$$
$$\max_{x_2,x_3,\ldots,x_n} M_{n,p}(x_2,x_3,\ldots,x_n) = \min\left\{\frac{2np}{n+1}, 1\right\}.$$

The upper bound increases with jury size n, while the lower bound decreases with n. Increasing n will increase the range of admissible Condorcet probabilities, but the drop in the lower bound becomes successively smaller as n increases. Higher individual competence increases the jury's competence, and the bounds increase in p.

Recall that x_2 is the Pearson product-moment correlation coefficient, a ubiquitous measure of stochastic dependence. For $x_2 > 0$, Condorcet's probability attains its unique minimum when $x_2 = \frac{2p-1}{2p}$. Since Condorcet's probability depends on coefficients of all orders, the downside risk associated with the unknown higher order dependencies is highest when the above relationship holds. Remarkably, the above relationship does not depend on n; it is valid for any jury comprising an odd number of jurors.

Can the collective competence of this jury fall below the 'competence' offered by a toss of a fair coin? For $p \geq \frac{3}{4}$, the lower bound is strictly larger than $\frac{1}{2}$. For $\frac{2}{3} < p < \frac{3}{4}$, the lower bound may exceed $\frac{1}{2}$ only for small n. For $p \leq \frac{2}{3}$, the lower bound cannot exceed $\frac{1}{2}$ for any n; it will equal $\frac{1}{2}$ only when $p = \frac{2}{3}$ and $n = 3$. This shows that a fair coin can outperform a large jury comprising not-too-competent jurors. The upper bound is always at least equal to $\frac{3}{4}$.

Our second theorem establishes that in a homogeneous jury the probability of casting a decisive vote is less than one, while it can be zero for infinitely many combinations of moments:

$$\min_{x_2,x_3,\ldots,x_n} P_{n,p}(x_2,x_3,\ldots,x_n) = 0;$$

$$\max_{x_2,x_3,\ldots,x_n} P_{n,p}(x_2,x_3,\ldots,x_n) = 2(1-p).$$

All other things equal, positive second-order correlation decreases the likelihood of ties, while negative correlation increases it. The intuition that power vanishes when the votes are perfectly positively correlated because ties cannot occur if every member votes the same way is correct, but perfect positive second-order correlation is not a necessary condition when higher-order correlations are present. There will be an infinite number of combinations of correlation coefficients of all orders that lead to zero power. In fact the smallest admissible Pearson correlation coefficient can be close to zero, which is remarkable in light of the fact that zero Pearson correlation is often mistakenly taken for stochastic independence. While this is true for Gaussian random variables, whose joint probability distribution function is completely characterized by the first two moments, zero second-order correlation between $n > 2$ Bernoulli random variables does not imply their independence. Higher order correlations matter.

Equally intuitively, the smaller the second-order correlation between the votes is, the higher the voting power should be. The intuition that power should increase as the correlation between the votes decreases is correct in principle, but its lower bound is size dependent.

We conclude by emphasizing that homogeneity is an assumption one would be reluctant to make in the presence of more specific information about the competence and dependence structure of the jury. Our results offer a benchmark against which the expertise of particular heterogeneous juries can be compared. The complexity of the optimization problem is such that for heterogeneous juries the bounds on Condorcet's probability should be computed numerically.

A Proofs

Proof of Theorem 1. Let us express Problems (7) and (8) in terms of $\lambda_2, \dots, \lambda_n$ ($\lambda_0 = 1, \lambda_1 = p$) using (2). The probabilities π_k and Condorcet's probability look as follows:

$$\pi_k = \sum_{j=0}^{k} (-1)^j C_k^j \lambda_{n-k+j}, \quad \text{for} \quad 0 \le k \le n-2;$$
$$\pi_{n-1} = p + \sum_{j=1}^{n-1} (-1)^j C_{n-1}^j \lambda_{j+1};$$
$$\pi_n = 1 - np + \sum_{j=2}^{n} (-1)^j C_n^j \lambda_j;$$

$$M_{n,p}(\lambda_2, \lambda_3, \dots, \lambda_n) = \sum_{k=0}^{\frac{n-1}{2}} C_n^k \sum_{j=0}^{k} (-1)^j C_k^j \lambda_{n-k+j} = \sum_{i=2}^{n} a_i \lambda_i,$$

where $a_2 = \dots = a_{\frac{n-1}{2}} = 0,\ a_{\frac{n+1}{2}+i} = (-1)^i C_n^{\frac{n-1}{2}-i} C_{\frac{n-1}{2}+i}^i,\ i = 0, 1, \dots, \frac{n-1}{2}$.

Therefore, in dual Problems (9) and (10), $\mathbf{b} = (0, \dots, 0, p, 1-np)^T,\ \mathbf{a} = (a_2, a_3, \dots, a_n)^T$,

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & C_{n-1}^1 & -C_n^2 \\ 0 & 0 & 0 & \dots & -1 & C_{n-2}^1 & -C_{n-1}^2 & C_n^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & -1 & \dots & -C_{n-3}^{n-5} & C_{n-2}^{n-4} & -C_{n-1}^{n-3} & C_n^{n-2} \\ 0 & -1 & 2 & \dots & C_{n-3}^{n-4} & -C_{n-2}^{n-3} & C_{n-1}^{n-2} & -C_n^{n-1} \\ -1 & 1 & -1 & \dots & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Consider the system of equations $\mathbf{A}^T \mathbf{u} = -\mathbf{a}$ from dual Problem (9). This system comprises $n-1$ equations and $n+1$ unknowns. By solving the above system of equations, the number of variables in Problem (9) is reduced from $n+1$ to 2, since

$$u_i = C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n + C_n^i, \quad \text{for} \quad i = 0, \ldots, \frac{n-1}{2};$$
$$u_i = C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n, \quad \text{for} \quad i = \frac{n+1}{2}, \ldots, n-2.$$

Therefore, the set of constraints $\mathbf{A}^T\mathbf{u} = -\mathbf{a}$, $u_i \geq 0$, $i = 0, 1, \ldots, n$, becomes

$$\begin{aligned} C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n + C_n^i &\geq 0, \quad \text{for} \quad i = 0, \ldots, \frac{n-1}{2}; \\ C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n &\geq 0, \quad \text{for} \quad i = \frac{n+1}{2}, \ldots, n-2; \\ u_{n-1} &\geq 0; \\ u_n &\geq 0. \end{aligned}$$

For the above constraints to hold, it suffices that $u_0 \geq 0$, $u_{\frac{n+1}{2}} \geq 0$, and $u_n \geq 0$. Consequently, the solutions of dual Problem (9) coincide with the solutions of the problem:

$$\mathbf{b}^T\mathbf{u} = pu_{n-1} + (1-np)u_n \to \min \quad \text{subject to} \quad \begin{aligned} & u_{n-1} - (n-1)u_n + 1 \geq 0; \\ & C_{n-1}^{\frac{n+1}{2}} u_{n-1} - \frac{n-3}{2} C_n^{\frac{n+1}{2}} u_n \geq 0; \\ & u_n \geq 0. \end{aligned} \tag{17}$$

It is easy to see that the solution (u_{n-1}^*, u_n^*) to the above problem does not depend on p and lies on the vertex defined by

$$u_{n-1} - (n-1)u_n + 1 = 0 \quad \text{and} \quad C_{n-1}^{\frac{n+1}{2}} u_{n-1} - \frac{n-3}{2} C_n^{\frac{n+1}{2}} u_n = 0.$$

Moreover, $pu^*_{n-1} + (1-np)u^*_n = \frac{n-1-2np}{n+1}$ at $(u^*_{n-1}, u^*_n) = (\frac{n(n-3)}{n+1}, \frac{n-1}{n+1})$. Thus, it follows from the solution of the dual problem that

$$\pi_0 \geq 0, \pi_1 = 0, \ldots, \pi_{\frac{n-1}{2}} = 0, \pi_{\frac{n+1}{2}} \geq 0, \pi_{\frac{n+3}{2}} = 0, \ldots, \pi_n = 0$$

and

$$\min_{x_2,x_3,\ldots,x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = \frac{n(2p-1)+1}{n+1}.$$

Problem (8) is solved in a similar manner. Switch to the solution of corresponding dual Problem (10) and consider the system of equations $\mathbf{A}^T\mathbf{u} = \mathbf{a}$. By solving this system, the number of variables in Problem (10) is also reduced from $n+1$ to 2, since

$$\begin{aligned} u_i &= C^i_{n-1}u_{n-1} - (n-1-i)C^i_n u_n - C^i_n, \quad \text{for} \quad i = 0, \ldots, \frac{n-1}{2}; \\ u_i &= C^i_{n-1}u_{n-1} - (n-1-i)C^i_n u_n, \quad \text{for} \quad i = \frac{n+1}{2}, \ldots, n-2. \end{aligned}$$

Therefore, the set of constraints $\mathbf{A}^T\mathbf{u} = \mathbf{a}$, $u_i \geq 0$, $i = 0, 1, \ldots, n$, reduces to

$$\begin{aligned} C^i_{n-1}u_{n-1} - (n-1-i)C^i_n u_n - C^i_n &\geq 0, \quad \text{for} \quad i = 0, \ldots, \frac{n-1}{2}; \\ C^i_{n-1}u_{n-1} - (n-1-i)C^i_n u_n &\geq 0, \quad \text{for} \quad i = \frac{n+1}{2}, \ldots, n-2; \\ u_{n-1} &\geq 0; \\ u_n &\geq 0. \end{aligned}$$

For the above constraints to hold, it suffices that $u_0 \geq 0$, $u_{\frac{n-1}{2}} \geq 0$, and $u_n \geq 0$. Consequently, the solutions of dual Problem (10) coincide with the solutions of the problem:

$$\mathbf{b}^T\mathbf{u} = pu_{n-1} + (1-np)u_n \to \min \quad \text{subject to} \quad \begin{aligned} & u_{n-1} - (n-1)u_n - 1 \geq 0; \\ & C_{n-1}^{\frac{n-1}{2}} u_{n-1} - \frac{n-1}{2} C_n^{\frac{n-1}{2}} u_n - C_n^{\frac{n-1}{2}} \geq 0; \\ & u_n \geq 0. \end{aligned} \tag{18}$$

The solution to the above problem (u_{n-1}^*, u_n^*) does depend on $p \in (0.5, 1)$. For $p < \frac{n+1}{2n}$ it corresponds to the vertex defined by

$$C_{n-1}^{\frac{n-1}{2}} u_{n-1} - \frac{n-1}{2} C_n^{\frac{n-1}{2}} u_n - C_n^{\frac{n-1}{2}} = 0 \quad \text{and} \quad u_n = 0.$$

In this case, $pu_{n-1}^* + (1-np)u_n^* = \frac{2np}{n+1}$ at $(u_{n-1}^*, u_n^*) = (\frac{2n}{n+1}, 0)$. Thus, it follows from the solution of the dual problem that

$$\pi_0 = 0, \ldots, \pi_{\frac{n-3}{2}} = 0, \pi_{\frac{n-1}{2}} \geq 0, \pi_{\frac{n+1}{2}} = 0, \ldots, \pi_{n-1} = 0, \pi_n \geq 0$$

and

$$\max_{x_2, x_3, \ldots, x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = \frac{2np}{n+1}.$$

For $p > \frac{n+1}{2n}$, the solution corresponds to the vertex defined by

$$u_{n-1} - (n-1)u_n - 1 = 0 \quad \text{and} \quad C_{n-1}^{\frac{n-1}{2}} u_{n-1} - \frac{n-1}{2} C_n^{\frac{n-1}{2}} u_n - C_n^{\frac{n-1}{2}} = 0.$$

In this case, $pu_{n-1}^* + (1-np)u_n^* = 1$ at $(u_{n-1}^*, u_n^*) = (n, 1)$. Now it follows from the solution of the dual problem that

$$\pi_0 \geq 0, \ldots, \pi_{\frac{n-1}{2}} \geq 0, \pi_{\frac{n+1}{2}} = 0, \ldots, \pi_n = 0$$

and

$$\max_{x_2, x_3, \ldots, x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = 1.$$

Finally, for $p = \frac{n+1}{2n}$ the solution of Problem (18) is non-unique; all points lying on the line $C_{n-1}^{\frac{n-1}{2}} u_{n-1} - \frac{n-1}{2} C_n^{\frac{n-1}{2}} u_n - C_n^{\frac{n-1}{2}} = 0$ between the points $(\frac{2n}{n+1}, 0)$ and $(n, 1)$ are the solutions of Problem (18). In this case,

$$\pi_0 = 0, \ldots, \pi_{\frac{n-3}{2}} = 0, \pi_{\frac{n-1}{2}} \geq 0 \text{ (and therefore } \pi_{\frac{n-1}{2}} = 1/C_n^{\frac{n-1}{2}}), \pi_{\frac{n+1}{2}} = 0, \ldots, \pi_n = 0$$

and

$$\max_{x_2, x_3, \ldots, x_n} M_{n,p}(x_2, x_3, \ldots, x_n) = 1.$$

□

Proof of Corollary 1. Rewriting the system of equations $\pi_1 = 0, \ldots, \pi_{\frac{n-1}{2}} = 0, \pi_{\frac{n+3}{2}} = 0, \ldots,$ $\pi_n = 0$ in terms of the variables $\lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n$, we obtain

$$\pi_k = \sum_{j=0}^{k-1} (-1)^j C_{k-1}^j [\lambda_{n-k+j} - \lambda_{n-k+j+1}] = 0, \quad \text{for} \quad 1 \leq k \leq n-2, k \neq \frac{n+1}{2}; \quad (19)$$

$$\pi_{n-1} = p - \lambda_2 + \sum_{j=1}^{n-2} (-1)^j C_{n-2}^j [\lambda_{j+1} - \lambda_{j+2}] = 0; \quad (20)$$

$$\pi_n = (1 - np) + (n-1)\lambda_2 + \sum_{j=2}^{n-1} (-1)^j C_{n-1}^j [\lambda_j - \lambda_{j+1}] = 0. \quad (21)$$

For $k = 1$, $\pi_1 = \lambda_{n-1} - \lambda_n = 0$, while for $k = 2$, $\pi_2 = (\lambda_{n-2} - \lambda_{n-1}) - (\lambda_{n-1} - \lambda_n) = 0$. Together, this implies that $\lambda_{n-2} = \lambda_{n-1} = \lambda_n$. Continuing in the same manner, we obtain $\lambda_{\frac{n+1}{2}} = \lambda_{\frac{n+3}{2}} = \cdots = \lambda_n$. In view of these equalities, System (19)-(21) can be expressed as:

$$\pi_k = \sum_{j=0}^{k-\frac{n+1}{2}} (-1)^j C_{k-1}^j [\lambda_{n-k+j} - \lambda_{n-k+j+1}] = 0, \quad \text{for} \quad \frac{n+3}{2} \leq k \leq n-2;$$

$$\pi_{n-1} = p - \lambda_2 + \sum_{j=1}^{\frac{n-3}{2}} (-1)^j C_{n-2}^j [\lambda_{j+1} - \lambda_{j+2}] = 0;$$

$$\pi_n = (1 - np) + (n-1)\lambda_2 + \sum_{j=2}^{\frac{n-1}{2}} (-1)^j C_{n-1}^j [\lambda_j - \lambda_{j+1}] = 0.$$

The above linear system with respect to $\lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{\frac{n-1}{2}} - \lambda_{\frac{n+1}{2}}$ is represented as an augmented matrix:

$$\left(\begin{array}{cccccc|c}
0 & \cdots & \cdots & 0 & 1 & -C^1_{\frac{n+1}{2}} & 0 \\
0 & \cdots & 0 & 1 & -C^1_{\frac{n+3}{2}} & C^2_{\frac{n+3}{2}} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 1 & -C^1_{n-4} & \cdots & (-1)^{\frac{n-7}{2}} C^{\frac{n-7}{2}}_{n-4} & 0 \\
0 & 1 & -C^1_{n-3} & \cdots & \cdots & (-1)^{\frac{n-5}{2}} C^{\frac{n-5}{2}}_{n-3} & 0 \\
-1 & -C^1_{n-2} & C^2_{n-2} & \cdots & \cdots & (-1)^{\frac{n-3}{2}} C^{\frac{n-3}{2}}_{n-2} & -p \\
C^1_{n-1} & C^2_{n-1} & -C^3_{n-1} & \cdots & \cdots & (-1)^{\frac{n-1}{2}} C^{\frac{n-1}{2}}_{n-1} & np-1
\end{array}\right).$$

Using elementary row operations we obtain:

$$\left(\begin{array}{cccccc|c}
0 & \dots & \dots & 0 & 0 & -C^{0}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & p-1 \\
0 & \dots & \dots & 0 & -C^{\frac{n-3}{2}}_{n-1} & C^{1}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & p-1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & -C^{4}_{n-1} & \dots & (-1)^{\frac{n-7}{2}} C^{3}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & p-1 \\
0 & 0 & -C^{3}_{n-1} & 3C^{4}_{n-1} & \dots & (-1)^{\frac{n-5}{2}} C^{2}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & p-1 \\
0 & -C^{2}_{n-1} & 2C^{3}_{n-1} & -3C^{4}_{n-1} & \dots & (-1)^{\frac{n-3}{2}} C^{1}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & p-1 \\
C^{1}_{n-1} & C^{2}_{n-1} & -C^{3}_{n-1} & C^{4}_{n-1} & \dots & (-1)^{\frac{n-1}{2}} C^{0}_{\frac{n-3}{2}} C^{\frac{n-1}{2}}_{n-1} & np-1
\end{array}\right).$$

Summing all rows, we get $\lambda_2 = \frac{3p-1}{2}$. Therefore,

$$x_2 = \frac{\lambda_2 - p^2}{p(1-p)} = \frac{2p-1}{2p}.$$

□

Proof of Theorem 2. Rewrite Problem (15) in the standard form as:

$$\mathbf{a}^T \mathbf{x} \to \max \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{22}$$

where $\mathbf{a} \in \mathbb{R}^{n-1}$, $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\mathbf{A} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$; and in the dual form as:

$$\mathbf{b}^T \mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T \mathbf{u} = \mathbf{a}, \quad u_i \geq 0, \quad i = 0, 1, \dots, n. \tag{23}$$

The matrix $\mathbf{A}$, the vectors $\mathbf{a}$ and $\mathbf{b}$ are defined as:

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -1 & C_{n-1}^1 & -C_n^2 \\ 0 & 0 & 0 & \dots & -1 & C_{n-2}^1 & -C_{n-1}^2 & C_n^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & -1 & \dots & C_{n-3}^{n-5} & -C_{n-2}^{n-4} & C_{n-1}^{n-3} & -C_n^{n-2} \\ 0 & -1 & 2 & \dots & -C_{n-3}^{n-4} & C_{n-2}^{n-3} & -C_{n-1}^{n-2} & C_n^{n-1} \\ -1 & 1 & -1 & \dots & 1 & -1 & 1 & -1 \end{pmatrix}.$$

$$\mathbf{a} = (0, \dots, 0, 1, -C_{n/2}^1, C_{n/2}^2, \dots, (-1)^{n/2} C_{n/2}^{n/2})^T \text{ and } \mathbf{b} = (0, \dots, 0, p, 1 - np)^T.$$

Consider the system of equations $\mathbf{A}^T\mathbf{u} = \mathbf{a}$. Since the solution of this system is given by

$$\begin{aligned} u_i &= C_{n-1}^i u_{n-1} - (n-1-i) C_n^i u_n, \quad \text{for} \quad i \neq \frac{n}{2},\ i \leq n-2; \\ u_{\frac{n}{2}} &= C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n - 1, \end{aligned}$$

the set of constraints $\mathbf{A}^T\mathbf{u} = \mathbf{a}$, $u_i \geq 0$, $i = 0, 1, \dots, n$, reduces to

$$\begin{aligned} C_{n-1}^i u_{n-1} - (n-1-i) C_n^i u_n &\geq 0, \quad \text{for} \quad i \neq \frac{n}{2},\ i \leq n-2; \\ C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n - 1 &\geq 0; \\ u_{n-1} &\geq 0; \\ u_n &\geq 0. \end{aligned}$$

For the above constraints to hold, it suffices that $u_0 \geq 0$, $u_{\frac{n}{2}} \geq 0$, and $u_n \geq 0$. Consequently, the solutions of dual Problem (23) coincide with the solutions of the problem:

$$\mathbf{b}^T\mathbf{u} = pu_{n-1} + (1-np)u_n \to \min \quad \text{subject to} \quad \begin{aligned} & u_{n-1} - (n-1)u_n \geq 0; \\ & C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n - 1 \geq 0; \\ & u_n \geq 0. \end{aligned} \tag{24}$$

The solution (u_{n-1}^*, u_n^*) to the above problem corresponds to the vertex defined by

$$C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n = 1 \quad \text{and} \quad u_{n-1} = (n-1)u_n,$$

that is $(u_{n-1}^*, u_n^*) = (\frac{n-1}{C_{n-1}^{\frac{n}{2}}}, \frac{1}{C_{n-1}^{\frac{n}{2}}})$, and $pu_{n-1}^* + (1-np)u_n^* = \frac{1-p}{C_{n-1}^{\frac{n}{2}}}$.

From the solution of the dual problem it now follows that

$$\pi_0 \geq 0, \pi_1 = 0, \ldots, \pi_{\frac{n}{2}-1} = 0, \pi_{\frac{n}{2}} > 0, \pi_{\frac{n}{2}+1} = 0, \ldots, \pi_n = 0.$$

Rewriting $\pi_1 = 0, \ldots, \pi_{\frac{n}{2}-1} = 0, \pi_{\frac{n}{2}+1} = 0, \ldots, \pi_n = 0$ in terms of $\lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n$, we obtain

$$\pi_k = \sum_{j=0}^{k-1} (-1)^j C_{k-1}^j [\lambda_{n-k+j} - \lambda_{n-k+j+1}] = 0, \quad \text{for} \quad 1 \leq k \leq n-2, k \neq \frac{n}{2}; \tag{25}$$

$$\pi_{n-1} = p - \lambda_2 + \sum_{j=1}^{n-2} (-1)^j C_{n-2}^j [\lambda_{j+1} - \lambda_{j+2}] = 0; \tag{26}$$

$$\pi_n = 1 - np + (n-1)\lambda_2 + \sum_{j=2}^{n-1} (-1)^j C_{n-1}^j [\lambda_j - \lambda_{j+1}] = 0. \tag{27}$$

For $k = 1$, $\pi_1 = \lambda_{n-1} - \lambda_n = 0$, while for $k = 2$, $\pi_2 = (\lambda_{n-2} - \lambda_{n-1}) - (\lambda_{n-1} - \lambda_n) = 0$. Together, this implies that $\lambda_{n-2} = \lambda_{n-1} = \lambda_n$. Continuing in the same manner, we obtain

$\lambda_{\frac{n}{2}+1} = \cdots = \lambda_n$. In view of these equalities, System (25)-(27) can be expressed as:

$$\pi_k = \sum_{j=0}^{k-\frac{n}{2}} (-1)^j C_{k-1}^j [\lambda_{n-k+j} - \lambda_{n-k+j+1}] = 0, \quad \text{for} \quad \frac{n}{2} + 1 \leq k \leq n-2;$$
$$\pi_{n-1} = p - \lambda_2 + \sum_{j=1}^{\frac{n}{2}-1} (-1)^j C_{n-2}^j [\lambda_{j+1} - \lambda_{j+2}] = 0;$$
$$\pi_n = 1 - np + (n-1)\lambda_2 + \sum_{j=2}^{\frac{n}{2}} (-1)^j C_{n-1}^j [\lambda_j - \lambda_{j+1}] = 0.$$

For $k = \frac{n}{2}$, we have

$$\pi_{\frac{n}{2}} = \lambda_{\frac{n}{2}} - \lambda_{\frac{n}{2}+1} = \frac{1-p}{C_{n-1}^{\frac{n}{2}}}.$$

For $k = \frac{n}{2} + 1$, we have

$$\pi_{\frac{n}{2}+1} = [\lambda_{\frac{n}{2}-1} - \lambda_{\frac{n}{2}}] - C_{\frac{n}{2}}^1 [\lambda_{\frac{n}{2}} - \lambda_{\frac{n}{2}+1}] = 0 \implies \lambda_{\frac{n}{2}-1} - \lambda_{\frac{n}{2}} = \frac{1-p}{C_{n-1}^{\frac{n}{2}}} C_{\frac{n}{2}}^1.$$

For $k = \frac{n}{2} + 2$, we have

$$\pi_{\frac{n}{2}+2} = [\lambda_{\frac{n}{2}-2} - \lambda_{\frac{n}{2}-1}] - C_{\frac{n}{2}+1}^1 [\lambda_{\frac{n}{2}-1} - \lambda_{\frac{n}{2}}] + C_{\frac{n}{2}+1}^2 [\lambda_{\frac{n}{2}} - \lambda_{\frac{n}{2}+1}] = 0 \implies$$

$$\lambda_{\frac{n}{2}-2} - \lambda_{\frac{n}{2}-1} = \frac{1-p}{C_{n-1}^{\frac{n}{2}}} \left[C_{\frac{n}{2}}^1 C_{\frac{n}{2}+1}^1 - C_{\frac{n}{2}+1}^2 \right] = \frac{1-p}{C_{n-1}^{\frac{n}{2}}} C_{\frac{n}{2}+1}^2 \tag{28}$$

and so on up to $k = n-2$, whence we obtain

$$\lambda_2 - \lambda_3 = \frac{1-p}{C_{n-1}^{\frac{n}{2}}} C_{n-3}^{\frac{n}{2}-2}. \tag{29}$$

The equation $\pi_{n-1} = 0$ implies

$$\begin{aligned}
\lambda_2 &= p - C^1_{n-2}[\lambda_2 - \lambda_3] + C^2_{n-2}[\lambda_3 - \lambda_4] - \ldots + (-1)^{\frac{n}{2}-1} C^{\frac{n}{2}-1}_{n-2}[\lambda_{\frac{n}{2}} - \lambda_{\frac{n}{2}+1}] \\
&= p - \frac{1-p}{C^{\frac{n}{2}}_{n-1}} \left[C^{\frac{n}{2}-2}_{n-3} C^1_{n-2} - C^{\frac{n}{2}-3}_{n-4} C^2_{n-2} + \ldots + (-1)^{\frac{n}{2}} C^0_{\frac{n}{2}-1} C^{\frac{n}{2}-1}_{n-2} \right] \\
&= p - \frac{1-p}{C^{\frac{n}{2}}_{n-1}} C^{\frac{n}{2}-1}_{n-2} = p - \frac{n(1-p)}{2(n-1)}.
\end{aligned}$$

We can now use (3) to obtain $x_2 = 1 - \frac{n}{2p(n-1)}$.

Finally, note that for $3 \le m \le \frac{n}{2} + 1$, summing equations (28)-(29) for $\lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \ldots, \lambda_{m-1} - \lambda_m$ yields

$$\begin{aligned}
\lambda_2 - \lambda_m &= \frac{1-p}{C^{\frac{n}{2}}_{n-1}} \left[C^{\frac{n}{2}-2}_{n-3} + C^{\frac{n}{2}-3}_{n-4} + \ldots + C^{\frac{n}{2}+1-m}_{n-m} \right] \Longrightarrow \\
\lambda_m &= p - \frac{1-p}{C^{\frac{n}{2}}_{n-1}} \left[C^{\frac{n}{2}-1}_{n-2} + C^{\frac{n}{2}-2}_{n-3} + C^{\frac{n}{2}-3}_{n-4} + \ldots + C^{\frac{n}{2}+1-m}_{n-m} \right] \Longrightarrow \\
\pi_0 &= \lambda_n = \lambda_{\frac{n}{2}+1} = p - \frac{1-p}{C^{\frac{n}{2}}_{n-1}} \left[C^{\frac{n}{2}-1}_{n-2} + C^{\frac{n}{2}-2}_{n-3} + C^{\frac{n}{2}-3}_{n-4} + \ldots + C^0_{\frac{n}{2}-1} \right] = 2p - 1.
\end{aligned}$$

Remark. For $0 < p < 0.5$, the solution of the maximum problem is slightly different:

$$\max_{x_2, x_3, \ldots, x_n} \pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n) = \frac{p}{C^{\frac{n}{2}}_{n-1}}.$$

$\square$

B Moments in the competence problem with three jurors

For $n = 3$ we have the following $k = 4$ distinct probabilities π_k:

$$\pi_0 = p^3 + 3x_2p^2(1-p) + x_3p^{\frac{3}{2}}(1-p)^{\frac{3}{2}};$$

$$\pi_1 = p^2(1-p) + x_2p(1-p)(1-3p) - x_3p^{\frac{3}{2}}(1-p)^{\frac{3}{2}};$$

$$\pi_2 = p(1-p)^2 + x_2p(1-p)(3p-2) + x_3p^{\frac{3}{2}}(1-p)^{\frac{3}{2}};$$

$$\pi_3 = (1-p)^3 + 3x_2p(1-p)^2 - x_3p^{\frac{3}{2}}(1-p)^{\frac{3}{2}},$$

and Condorcet's probability

$$M_{3,p}(x_2, x_3) = \pi_0 + 3\pi_1 = p^2(3-2p) - 3x_2p(1-p)(2p-1) - 2x_3p^{\frac{3}{2}}(1-p)^{\frac{3}{2}}.$$

In terms of $\lambda_2 = P(V_1 = 1, V_2 = 1)$ and $\lambda_3 = P(V_1 = 1, V_2 = 1, V_3 = 1)$ the above probabilities look as follows:

$$\pi_0 = \lambda_3;$$

$$\pi_1 = \lambda_2 - \lambda_3;$$

$$\pi_2 = p - 2\lambda_2 + \lambda_3;$$

$$\pi_3 = 1 - 3p + 3\lambda_2 - \lambda_3;$$

$$M_{3,p}(\lambda_2, \lambda_3) = 3\lambda_2 - 2\lambda_3.$$

Our aim is to solve the following linear programming problems:

$$M_{3,p}(\lambda_2, \lambda_3) \to \min_{\lambda_2,\lambda_3}, \ M_{3,p}(\lambda_2, \lambda_3) \to \max_{\lambda_2,\lambda_3} \quad \text{subject to} \quad \pi_0 \geq 0, \pi_1 \geq 0, \pi_2 \geq 0, \pi_3 \geq 0.$$

Consider the minimum problem. The corresponding dual problem has the form:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = -\mathbf{a}, \quad u_i \geq 0,$$

where $\mathbf{b}^T = (0, 0, p, 1-3p)$, $\mathbf{A}^T = \begin{pmatrix} 0 & -1 & 2 & -3 \\ -1 & 1 & -1 & 1 \end{pmatrix}$, $-\mathbf{a} = (-3, 2)^T$.

Solving the system of equations $\mathbf{A}^T\mathbf{u} = -\mathbf{a}$:

$$\left(\begin{array}{cccc|c} 0 & -1 & 2 & -3 & -3 \\ -1 & 1 & -1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & -1 & 2 & -3 & -3 \\ -1 & 0 & 1 & -2 & -1 \end{array}\right),$$

we obtain $u_0 = u_2 - 2u_3 + 1$ and $u_1 = 2u_2 - 3u_3 + 3$. Now our dual problem reads:

$$pu_2 + (1-3p)u_3 \to \min \quad \text{subject to} \quad u_2 - 2u_3 + 1 \geq 0, 2u_2 - 3u_3 + 3 \geq 0, u_2 \geq 0, u_3 \geq 0.$$

The solution of this problem $(u_2^*, u_3^*) = (0, 0.5)$ does not depend on p.

Let us return to the primal problem. From the solution of the dual problem it follows that $\min M_{3,p} = -pu_2^* - (1-3p)u_3^* = \frac{3p-1}{2}$, and $\pi_0 \geq 0$, $\pi_1 = 0$, $\pi_2 \geq 0$, $\pi_3 = 0$.

Solving the system of linear equations $\pi_1 = 0$ and $\pi_3 = 0$, we obtain:

$$\lambda_2^* = \lambda_3^* = \frac{3p-1}{2}, \quad \text{or} \quad x_2^* = \frac{\lambda_2^* - p^2}{p(1-p)} = \frac{2p-1}{2p}, \quad x_3^* = \frac{\lambda_3^* - 3p\lambda_2^* + 2p^3}{p^{\frac{3}{2}}(1-p)^{\frac{3}{2}}} = \frac{(4p-1)(1-p)^{\frac{1}{2}}}{2p^{\frac{3}{2}}}.$$

The maximum problem is approached in a similar manner. Take the dual problem:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = \mathbf{a}, \quad u_i \geq 0.$$

Solving the system of equations $\mathbf{A}^T\mathbf{u} = \mathbf{a}$:

$$\left(\begin{array}{cccc|c} 0 & -1 & 2 & -3 & 3 \\ -1 & 1 & -1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & -1 & 2 & -3 & 3 \\ -1 & 0 & 1 & -2 & 1 \end{array}\right),$$

we obtain $u_0 = u_2 - 2u_3 - 1$ and $u_1 = 2u_2 - 3u_3 - 3$. Now our dual problem reads:

$$pu_2 + (1-3p)u_3 \to \min \quad \text{subject to} \quad u_2 - 2u_3 - 1 \geq 0, 2u_2 - 3u_3 - 3 \geq 0, u_2 \geq 0, u_3 \geq 0.$$

The solution (u_2^*, u_3^*) of this problem depends on p:

$$(u_2^*, u_3^*) = \begin{cases} (1.5, 0), & \text{if } \; p < \frac{2}{3} \\ (1.5(1+a), a), & \text{if } \; p = \frac{2}{3} \\ (3, 1), & \text{if } \; p > \frac{2}{3} \end{cases}$$

where a is any real number from the interval $[0, 1]$.

Returning to the primal problem, the solution of the dual problem implies that $\max M_{3,p} = pu_2^* + (1-3p)u_3^* = \min\{1.5p, 1\}$, and

$$\begin{aligned} \pi_0 = 0,\ \pi_1 \geq 0,\ \pi_2 = 0,\ \pi_3 \geq 0 \quad &\text{for} \quad p < \frac{2}{3}; \\ \pi_0 = 0,\ \pi_1 = \frac{1}{3},\ \pi_2 = 0,\ \pi_3 = 0 \quad &\text{for} \quad p = \frac{2}{3}; \\ \pi_0 \geq 0,\ \pi_1 \geq 0,\ \pi_2 = 0,\ \pi_3 = 0 \quad &\text{for} \quad p > \frac{2}{3}. \end{aligned}$$

Solving the system of linear equations $\pi_0 = 0$ and $\pi_2 = 0$ we obtain:

$$\lambda_2^* = \frac{p}{2}, \quad \lambda_3^* = 0, \quad \text{or} \quad x_2^* = -\frac{2p-1}{2(1-p)}, \quad x_3^* = \frac{(4p-3)p^{\frac{1}{2}}}{2(1-p)^{\frac{3}{2}}},$$

while solving the system of linear equations $\pi_2 = 0$ and $\pi_3 = 0$ yields

$$\lambda_2^* = 2p - 1, \quad \lambda_3^* = 3p - 2, \quad \text{or} \quad x_2^* = -\frac{1-p}{p}, \quad x_3^* = -\frac{2(1-p)^{\frac{3}{2}}}{p^{\frac{3}{2}}}.$$

For $p = \frac{2}{3}$ we have

$$\lambda_2^* = \frac{1}{3}, \quad \lambda_3^* = 0, \quad \text{or} \quad x_2^* = -\frac{1}{2}, \quad x_3^* = -\frac{1}{\sqrt{2}}.$$

C Moments in the voting power problem with five voters

For $n = 4$ (total number of voters is $n+1$) we have the following $k = 5$ distinct probabilities π_k:

$$\pi_0 = \lambda_4;$$

$$\pi_1 = \lambda_3 - \lambda_4;$$

$$\pi_2 = \lambda_2 - 2\lambda_3 + \lambda_4;$$

$$\pi_3 = p - 3\lambda_2 + 3\lambda_3 - \lambda_4;$$

$$\pi_4 = 1 - 4p + 6\lambda_2 - 4\lambda_3 + \lambda_4.$$

Our aim is to solve the following linear programming problems:

$$\pi_2(\lambda_2, \lambda_3, \lambda_4) \to \min_{\lambda_2,\lambda_3,\lambda_4}, \; \pi_2(\lambda_2, \lambda_3, \lambda_4) \to \max_{\lambda_2,\lambda_3,\lambda_4} \quad \text{subject to} \quad \pi_0 \geq 0, \pi_1 \geq 0, \pi_2 \geq 0, \pi_3 \geq 0, \pi_4 \geq 0.$$

Consider the minimum problem. The corresponding dual problem has the form:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = -\mathbf{a}, \quad u_i \geq 0,$$

where $\mathbf{b}^T = (0,0,0,p,1-4p)$, $\mathbf{A}^T = \begin{pmatrix} 0 & 0 & -1 & 3 & -6 \\ 0 & -1 & 2 & -3 & 4 \\ -1 & 1 & -1 & 1 & -1 \end{pmatrix}$, $-\mathbf{a} = (-1,2,-1)^T$.

Solving the system of equations $\mathbf{A}^T\mathbf{u} = -\mathbf{a}$:

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & 3 & -6 & -1 \\ 0 & -1 & 2 & -3 & 4 & 2 \\ -1 & 1 & -1 & 1 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 0 & 0 & -1 & 3 & -6 & -1 \\ 0 & -1 & 0 & 3 & -8 & 0 \\ -1 & 0 & 0 & 1 & -3 & 0 \end{array}\right),$$

we obtain $u_0 = u_3 - 3u_4$, $u_1 = 3u_3 - 8u_4$ and $u_2 = 3u_3 - 6u_4 + 1$. Now our dual problem reads:

$$pu_3 + (1-4p)u_4 \to \min \quad \text{subject to} \quad u_3 - 3u_4 \geq 0, 3u_3 - 8u_4 \geq 0, 3u_3 - 6u_4 + 1 \geq 0, u_3 \geq 0, u_4 \geq 0.$$

The solution of this problem is $(u_3^*, u_4^*) = (0,0)$.

Let us return to the primal problem. From the solution of the dual problem it follows that $\min \pi_2 = -pu_3^* - (1-4p)u_4^* = 0$, and $\pi_0 \geq 0$, $\pi_1 \geq 0$, $\pi_2 = 0$, $\pi_3 \geq 0$, $\pi_4 \geq 0$.

The maximum problem is approached in a similar manner. Take the dual problem:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = \mathbf{a}, \quad u_i \geq 0.$$

Solving the system of equations $\mathbf{A}^T\mathbf{u} = \mathbf{a}$:

$$\left(\begin{array}{ccccc|c} 0 & 0 & -1 & 3 & -6 & 1 \\ 0 & -1 & 2 & -3 & 4 & -2 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 0 & 0 & -1 & 3 & -6 & 1 \\ 0 & -1 & 0 & 3 & -8 & 0 \\ -1 & 0 & 0 & 1 & -3 & 0 \end{array}\right),$$

we obtain $u_0 = u_3 - 3u_4,\ u_1 = 3u_3 - 8u_4$ and $u_2 = 3u_3 - 6u_4 - 1$. Now our dual problem reads:

$$pu_3 + (1-4p)u_4 \to \min \quad \text{subject to} \quad u_3 - 3u_4 \geq 0, 3u_3 - 8u_4 \geq 0, 3u_3 - 6u_4 - 1 \geq 0, u_3 \geq 0, u_4 \geq 0.$$

The solution (u_3^*, u_4^*) of this problem is $(1, \frac{1}{3})$.

Returning to the primal problem, the solution of the dual problem implies that $\max \pi_2 = pu_3^* + (1-4p)u_4^* = \frac{1-p}{3}$, and $\pi_0 \geq 0,\ \pi_1 = 0,\ \pi_2 > 0,\ \pi_3 = 0,\ \pi_4 = 0$.

Solving the system of linear equations $\pi_1 = 0,\ \pi_3 = 0$ and $\pi_4 = 0$ we obtain:

$$\lambda_2^* = \frac{5p-2}{3}, \quad \lambda_3^* = \lambda_4^* = 2p - 1, \quad \text{and} \quad \pi_0 = 2p - 1.$$

From this we obtain

$$x_2^* = \frac{\lambda_2^* - p^2}{p(1-p)} = 1 - \frac{2}{3p}, \quad x_3^* = \frac{\lambda_3^* - 3p\lambda_2^* + 2p^3}{p^{3/2}(1-p)^{3/2}} = \frac{(2p-1)(1-p)^{1/2}}{p^{3/2}} = \left(2 - \frac{1}{p}\right)\left(\frac{1}{p} - 1\right)^{1/2},$$

$$x_4^* = \frac{\lambda_4^* - 4p\lambda_3^* + 6p^2\lambda_2^* - 3p^4}{p^2(1-p)^2} = \frac{(3p-1)(1-p)}{p^2} = \left(3 - \frac{1}{p}\right)\left(\frac{1}{p} - 1\right).$$

D *** Supplementary material for referee use only ***

The minimum of voting power. A theorem analogous to Theorem 2 can also be proven for the minimum. Written in the standard form Problem (14) reads

$$\mathbf{a}^T\mathbf{x} \to \min \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{30}$$

where $\mathbf{a} \in \mathbb{R}^{n-1}$, $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\mathbf{A} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$; and in dual form as:

$$\mathbf{b}^T\mathbf{u} \to \min \quad \text{subject to} \quad \mathbf{A}^T\mathbf{u} = -\mathbf{a}, \quad u_i \geq 0, \quad i = 0, 1, \ldots, n. \tag{31}$$

Here $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{A}$ are defined as in Theorem 2.

Consider the system of equations $\mathbf{A}^T\mathbf{u} = -\mathbf{a}$ from dual Problem (31). Using

$$\begin{aligned} u_i &= C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n, \quad \text{for} \quad i \neq \frac{n}{2},\ i \leq n-2; \\ u_{\frac{n}{2}} &= C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n + 1, \end{aligned}$$

we can rewrite the set of constraints $\mathbf{A}^T\mathbf{u} = -\mathbf{a}$, $u_i \geq 0$, $i = 0, 1, \ldots, n$, as

$$\begin{aligned} C_{n-1}^i u_{n-1} - (n-1-i)C_n^i u_n &\geq 0, \quad \text{for} \quad i \neq \frac{n}{2},\ i \leq n-2; \\ C_{n-1}^{\frac{n}{2}} u_{n-1} - \left(\frac{n}{2} - 1\right) C_n^{\frac{n}{2}} u_n + 1 &\geq 0; \\ u_{n-1} &\geq 0; \\ u_n &\geq 0. \end{aligned}$$

For the above constraints to hold, it suffices that $u_0 \geq 0$, $u_n \geq 0$. Consequently, the solutions of dual Problem (31) coincide with the solutions of the problem:

$$\mathbf{b}^T\mathbf{u} = pu_{n-1} + (1-np)u_n \to \min \quad \text{subject to} \quad u_{n-1} - (n-1)u_n \geq 0,\ u_n \geq 0. \qquad (32)$$

It is easy to show that the solution (u^*_{n-1}, u^*_n) to the above problem is $(0,0)$. Moreover, $pu^*_{n-1} + (1-np)u^*_n = 0$ at $(u^*_{n-1}, u^*_n) = (0,0)$. It follows from the solution of the dual problem that

$$\pi_0 \geq 0, \pi_1 \geq 0, \ldots, \pi_{\frac{n}{2}-1} \geq 0, \pi_{\frac{n}{2}} = 0, \pi_{\frac{n}{2}+1} \geq 0, \ldots, \pi_n \geq 0$$

and

$$\min_{x_2,x_3,\ldots,x_n} P_{n,p}(x_2, x_3, \ldots, x_n) = \min_{x_2,x_3,\ldots,x_n} \pi_{\frac{n}{2}}(x_2, x_3, \ldots, x_n) = 0.$$

□

This solution is also valid for $0 < p < 0.5$.

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