An Invariance Result for Homogeneous Juries with Correlated Votes

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Abstract

A joint probability distribution on the set of voting profiles is called second-order invariant if the probability of a jury collectively making the correct decision under simple majority rule (Condorcet's probability) is independent of second-order correlations. This paper establishes the existence of such distributions for homogeneous juries of an arbitrary size. In a homogeneous jury each juror's vote has an equal probability of being correct, and each pair of jurors' votes correlates with the same correlation coefficient.

JEL-Codes: C63, D72
Key Words: Condorcet's Jury Theorem, correlated votes, homogeneous jury

1 Introduction

In its classic version, Condorcet's jury theorem (CJT) assumes that a jury decides between two alternatives by voting under simple majority rule, each juror has more than an even probability of being correct, all jurors have equal probabilities, and each juror makes his decision independently. The theorem states that any jury comprising an odd number of jurors is more likely than any single juror to select the correct alternative, and that this likelihood becomes a certainty as the number of jurors tends to infinity. The probability of the jury collectively making the correct decision by voting under simple majority rule is called *Condorcet's probability*.

Empirical evidence shows that votes typically are positively correlated. This is true of the U.S. Supreme Court (Kaniovski and Leech 2009) and the Supreme Court of Canada (Heard and Swartz 1998), and of non-judicial voting bodies such as the European Union Council of Ministers (Hayes-Renshaw, van Aken and Wallace 2006) and the institutions of the United Nations (Newcombe, Ross and Newcombe 1970). This evidence substantiates the large body of research seeking to relax the independence assumption.

The early extensions of CJT to correlated votes assume sequential voting. In Boland (1989) and Boland, Proschan and Tong (1989) the course of voting is shaped by an opinion leader's vote. Berg (1993a, 1993b) models voting as an urn process, in which partial sums of votes

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are formed by indicator random variables with path dependent probabilities. Ladha (1992) formalizes a juror's competence as a probability conditional on the juror's information, and obtains sufficient conditions for both parts of the theorem in terms of an upper bound on the average of positive correlation coefficients. Ladha (1993, 1995) obtains stronger results for several particular distributional assumptions. Despite each model extending the probabilistic setting of the theorem in a unique way, they commonly conclude that CJT remains valid for low correlation, the upper bound being specific to each model. In contrast, Berend and Sapir (2007) provide a general non-asymptotic result in terms of a recurrence relation involving the average competence of a randomly chosen group of more than three jurors.

I assume that the competence and dependence of a jury is specified by the probability of a juror's vote being correct (marginal probability) and the Pearson product-moment correlation coefficient between two correct votes as the simplest and most widely used measure of stochastic dependence. The Pearson product-moment correlation coefficient is a pairwise or second-order measure. In a *homogeneous* jury each juror's vote has an equal probability of being correct, and each pair of jurors' votes correlates with the same correlation coefficient. Note that the classic version of CJT assumes homogeneity.

This paper is motivated by a numerical example in Ladha (1992) and its parametric generalization in Berend and Sapir (2007). The example shows a homogeneous jury comprising three jurors, in which the votes are uncorrelated so that Condorcet's probability is invariant to the correlation coefficient, yet the juror's votes are not independent (Section 2). The distribution in the example is second-order invariant in the sense that Condorcet's probability does not change with the second-order correlation coefficient. The fact that the number of jurors is fixed and the coefficient is zero makes the example a special case. This paper establishes the existence of second-order invariant distributions for homogeneous juries of an arbitrary size, in which the coefficient can assume a continuum of values, including negative values (Section 3). Numerical examples are provided in Section 4.

This paper concludes that the robustness of CJT to correlation cannot be verified based on second-order correlations only, as the marginal probabilities and second-order correlation coefficients do not uniquely define a joint probability distribution on the set of voting profiles, and there exist distributions for which Condorcet's probability is invariant to the second-order correlation coefficient. We must thus look at higher-order correlations or the joint probability distribution, the estimation of which to a reasonable degree of accuracy requires extensive voting data.

Homogeneity implies a representative agent, but the quadratic optimization problem used to obtain the main result allows varying marginal probabilities and second-order correlation coefficients. While in its full generality the optimization problem can only be solved numerically, an analytical solution for distributions is provided in which all the marginal probabilities are identical, but the correlation coefficients may vary (Appendix A).

2 The motivating example

Berend and Sapir (2007) construct the following example of a homogeneous jury comprising three jurors ($n = 3$). Let the probability of a juror voting for the correct alternative be $p_i = 1 - \sqrt{\frac{a}{3}}$ for $i = 1, 2, 3$.[1] A voting profile is represented by a binary vector $\mathbf{v} = (v_1, v_2, v_3)$, whose i-th coordinate $v_i = 1$ if juror i is correct, and equals 0 otherwise. The joint probability distribution

[1]Since the jurors are competent, $0 < a < 0.75$. In Ladha's (1992) simpler example $a = 0.6075$.

$\pi_{\mathbf{v}}$ on the set of eight conceivable voting profiles is as follows

#	v_1	v_2	v_3	$\pi_{\mathbf{v}}$
I	1	1	1	$1+a-\sqrt{3a}$
II	1	1	0	$(\sqrt{3a}-2a)/3$
III	1	0	1	$(\sqrt{3a}-2a)/3$
IV	1	0	0	$a/3$
V	0	1	1	$(\sqrt{3a}-2a)/3$
VI	0	1	0	$a/3$
VII	0	0	1	$a/3$
VIII	0	0	0	0

For two Bernoulli random variables[2] V_i, V_j with $E(V_i) = p_i$, $E(V_j) = p_j$, the second-order correlation coefficient is given by

$$c_{i,j} = \frac{\mathrm{Cov}(V_i, V_j)}{\sqrt{\mathrm{Var}(V_i)\mathrm{Var}(V_j)}} = \frac{P\{V_i = 1, V_j = 1\} - p_i p_j}{\sqrt{p_i q_i p_j q_j}} \quad \text{where} \quad q_i = 1 - p_i. \tag{1}$$

In the above example, any two votes are uncorrelated, as

$$\mathrm{Cov}(V_i, V_j) = 1 + a - \sqrt{3a} + \frac{1}{3}\left(\sqrt{3a} - 2a\right) - \left(1 - \sqrt{\frac{a}{3}}\right)^2 = 0 \quad \forall i \neq j.$$

However, the jurors' votes are not independent, as each juror has a positive probability of being incorrect. At the same time, the probability of all three jurors being simultaneously incorrect equals zero.

An alternative way of verifying the dependence is to show that a higher-order correlation coefficient is different from zero. Higher-order correlation coefficients measure dependence between k-tuples of votes. Higher-order correlation coefficients can be computed as follows. Let $Z_i = (V_i - p_i)/\sqrt{p_i q_i}$, then

$$\begin{aligned}
&c_{i,j} = E(Z_i Z_j) \quad \forall 1 \le i < j \le n;\\
&c_{i,j,k} = E(Z_i Z_j Z_k) \quad \forall 1 \le i < j < k \le n;\\
&\dots\\
&c_{1,2,\dots,n} = E(Z_1 Z_2 \dots Z_n).
\end{aligned}$$

In the above example, the third-order correlation coefficient is positive unless $a = 0$

$$c_{1,2,3} = \left(\sqrt{\frac{a}{3}}\right)^{\frac{3}{2}} \left(1 - \sqrt{\frac{a}{3}}\right)^{-\frac{3}{2}}. \tag{2}$$

With n jurors there will be $\sum_{i=2}^{n} C_n^i = 2^n - n - 1$ correlation coefficients, of which C_n^2 are second-order.[3] Bahadur (1961) shows that $2^n - n - 1$ correlation coefficients together with n marginal probabilities uniquely define the joint probability distribution of n Bernoulli random variables, and that n such variables are independent if and only if all correlation coefficients vanish.

[2] Upper-case letters denote random variables, and lower-case letters denote realizations.

[3] Here and below C_n^x denotes the binomial coefficient $C_n^x = \frac{n!}{x!(n-x)!}$ for $n, x \in \mathbb{N}$, where $C_n^x = 0$ for $n < x$.

The above example reminds us that zero second-order correlations between three or more Bernoulli random variables do not imply their stochastic independence, despite the fact that two uncorrelated Bernoulli random variables are independent. It also shows that, in the empirically relevant case of a dependence structure, specified only up to second-order correlation coefficients, the validity of CJT cannot be generally attested. In the next section I construct a distribution for which Condorcet's probability for a homogeneous jury is invariant to the common second-order correlation coefficient. The jury can be of an arbitrary size and the correlation coefficient can assume a continuum of values.

3 The model

In a jury comprising an odd number of jurors n, let p_i denote the probability of the i-th juror voting for the correct alternative and $c_{i,j}$ denote the Pearson product-moment correlation coefficient between any two such votes. The jurors are assumed to be competent, so that $p_i > 0.5$ for all $i = 1, 2, \ldots, n$. In the classic version of CJT, $p_i = p$ ($p > 0.5$) for all $i = 1, 2, \ldots, n$, and $c_{i,j} = 0$ for all $1 \leq i < j \leq n$. In a general homogeneous jury, $p_i = p$ and $c_{i,j} = c$ ($c \neq 0$).

A voting profile is a binary vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, whose i-th coordinate $v_i = 1$ if juror i votes for the correct alternative, and equals 0 otherwise. Let $\mathbf{V}$ be the set of all voting profiles, $\mathbf{V}(i)$ the set of voting profiles in which juror i votes for the correct alternative, i.e. the set of all binary vectors $\mathbf{v}$ such that $v_i = 1$, and $\mathbf{V}(i,j) = \mathbf{V}(i) \cap \mathbf{V}(j)$ the set of voting profiles in which jurors i and j both vote for the correct alternative, i.e. the set of all binary vectors $\mathbf{v}$ such that $v_i = v_j = 1$. The sets $\mathbf{V}$, $\mathbf{V}(i)$ and $\mathbf{V}(i,j)$ respectively contain 2^n, 2^{n-1} and 2^{n-2} elements. In the example in the previous section, the set $\mathbf{V}$ contains all eight vectors, the set $\mathbf{V}(2)$ contains vectors I, II, V, and VI, as only they have 1 in the second coordinate, and the set $\mathbf{V}(2,3)$ contains vectors I and V, as only they have 1 in the second and third coordinates.

A joint probability distribution $\pi_{\mathbf{v}}$ on the set of voting profiles $\mathbf{V}$ which satisfies given marginal probabilities and correlation coefficients must satisfy the following constraints:

$$\pi_{\mathbf{v}} \geq 0 \quad \forall \mathbf{v} \in \mathbf{V}; \tag{3}$$
$$\sum_{\mathbf{v} \in \mathbf{V}} \pi_{\mathbf{v}} = 1; \tag{4}$$
$$\sum_{\mathbf{v} \in \mathbf{V}(i)} \pi_{\mathbf{v}} = p_i \quad \forall i = 1, 2, \ldots, n \quad \text{(marginal probabilities)}; \tag{5}$$
$$\sum_{\mathbf{v} \in \mathbf{V}(i,j)} \pi_{\mathbf{v}} = p_i p_j + c_{i,j}\sqrt{p_i q_i p_j q_j} \quad \forall q_i = 1 - p_i, \quad 1 \leq i < j \leq n, \tag{6}$$

where the last equality follows from (1).

Constraints (3)-(6) define a convex polytope $\Delta \subset \mathbb{R}^{2^n}$. Any point in Δ is a suitable distribution. Such a distribution does not exist when $\Delta = \emptyset$, i.e. when the constraints are inconsistent. Typically this set is not empty and contains an infinite number of distributions satisfying given marginal probabilities and correlation coefficients. Indeed, the system (4)-(6) comprising $1 + n + C_n^2$ equations for 2^n unknowns typically has an infinite number of solutions for $n \geq 3$.

A particular solution can be selected by imposing an additional criterium, usually a minimum point in Δ of a strictly convex function $f : \mathbb{R}^{2^n} \mapsto \mathbb{R}$. The technique of imposing additional criteria on the solution in order for it to have certain properties, in our case uniqueness, is called regularization.

The following optimization problem is designed to select a distribution which is closest in the sense of least square deviations to the distribution in the case of independent votes:

$$\min_{\pi_{\mathbf{v}}} \quad 0.5 \sum_{\mathbf{v}} \left[\pi_{\mathbf{v}} - \prod_{i=1}^{n} p_i^{v_i} q_i^{(1-v_i)} \right]^2 . \tag{7}$$

With the full set of constraints imposed, the above quadratic optimization problem can only be solved numerically. Provided we confine attention to distribution with positive coordinates, a slightly less general version of the optimization problem, in which all marginal probabilities are identical but correlation coefficients may vary, can be solved analytically using the Lagrange multiplier method (Appendix A). In the special case of a homogeneous voting body in which $p_i = p$, and $c_{i,j} = c$, the solution can be expressed in terms of the number of votes in favor of the correct alternative $t_{\mathbf{v}} = \sum_{i=1}^{n} v_i$, as

$$\pi^*_{\mathbf{v}} = p^{t_{\mathbf{v}}} q^{n-t_{\mathbf{v}}} + 2^{2-n} pqc \Big[0.5n(n-1) + 2t_{\mathbf{v}}(t_{\mathbf{v}} - n) \Big]. \tag{8}$$

In the terminology of nonlinear constrained optimization, the analytical solution ignores the complementary slackness conditions imposed by (3), so that its coordinates will become negative for a sufficiently large $|c|$. This puts an upper bound on $|c|$ for given n and p. To obtain a rough upper-bound, note that the function $f(x) = 0.5n(n-1) + 2x(x-n)$ for $x = 0, 1, \dots, n$ is minimized for $x_1 = 0.5(n-1)$, and $x_2 = 0.5(n+1)$, and maximized for $x_3 = 0$, and $x_4 = n$. Moreover, $f(x_1) = f(x_2)$ and $f(x_3) = f(x_4)$. Then, $\pi^*_{\mathbf{v}} \geq 0$ if

$$0 < c \leq \frac{2^{n-1}(pq)^{\frac{n-1}{2}}}{q(n-1)} \quad \text{for} \quad t_{\mathbf{v}} = 0.5(n-1); \tag{9}$$

$$0 < c \leq \frac{2^{n-1}(pq)^{\frac{n-1}{2}}}{p(n-1)} \quad \text{for} \quad t_{\mathbf{v}} = 0.5(n+1); \tag{10}$$

$$-\frac{(2q)^{n-1}}{pn(n-1)} \leq c < 0 \quad \text{for} \quad t_{\mathbf{v}} = 0; \tag{11}$$

$$-\frac{(2p)^{n-1}}{qn(n-1)} \leq c < 0 \quad \text{for} \quad t_{\mathbf{v}} = n. \tag{12}$$

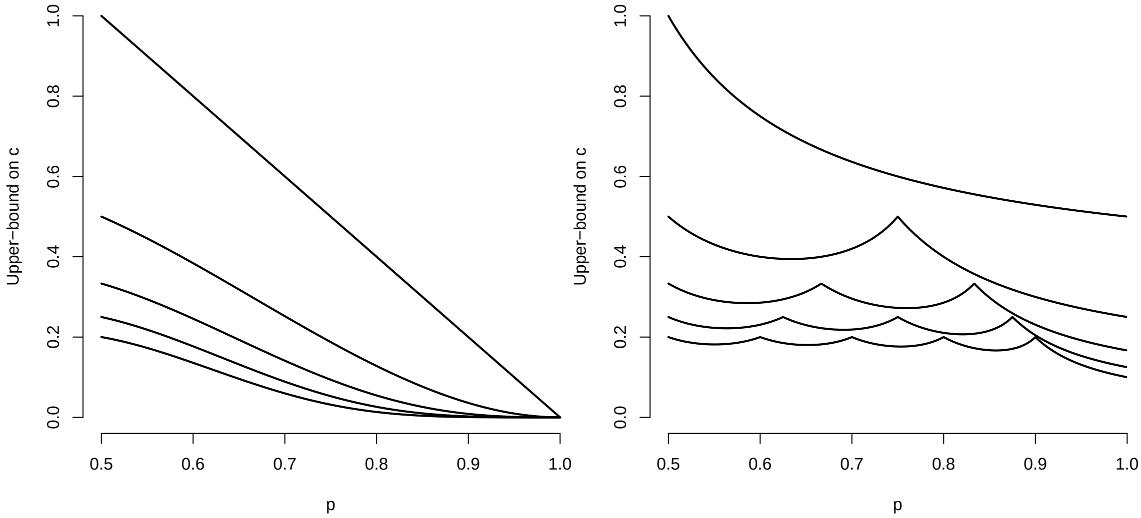
Since $p > q$ by the assumption of individual competence,

$$-\frac{(2q)^{n-1}}{pn(n-1)} \leq c \leq \frac{2^{n-1}(pq)^{\frac{n-1}{2}}}{p(n-1)}. \tag{13}$$

The upper bound on $|c|$ is tighter for $c < 0$ than for $c > 0$. The nature of the inequalities is such that for a given p, $|c| \to 0$ as $n \to \infty$, and for a given n, $|c| \to 0$ as $p \to 1$.

Two remarks are in order before proceeding to the statement and proof of the main result. First, $c = 1$ fulfills the inequality (13) only for $p = 0.5$ and $n = 3$, while $c = -1$ does not for any n. A joint probability distribution does not exist when the votes are perfectly negatively correlated. If one juror is correct whenever another is incorrect, and vice versa, then by virtue of binary choice a third juror cannot be simultaneously discordant with the former two. Three jurors cannot be mutually contrarian. By contrast, positive correlations reflect common rather than contrarian tendencies, making high consensus voting outcomes more probable than they would be if the jurors were independent.

Figure 1: The upper bound on $c > 0$



The left panel shows upper bounds on $c > 0$ for $n = 3$, $n = 5$, $n = 7$, $n = 9$ and $n = 11$. The right panel shows the analogous upper bounds for distributions in which all higher-order correlation coefficients are zero (Bahadur 1961). All upper bounds decrease with n and p. For a given n, the two types of upper bounds coincide when $p = 0.5$.

Second, the above upper bound on the second-order correlation coefficient is specific to the solution (8), the distribution closest in the sense of least square deviations to the distribution in the case of independent votes. Other distributions belonging to Δ, the set of all distributions satisfying constraints (3)-(6), may have different upper bounds.[4] Bahadur (1961) provides the upper bound for a distribution in which all higher-order correlations are zero:

$$-\frac{2q}{pn(n-1)} \leq c_B \leq \frac{2pq}{(n-1)pq + 0.25 - \gamma}, \tag{14}$$

where $\gamma = \min_{t_{\mathbf{v}}}\{[t_{\mathbf{v}} - (n-1)p - 0.5]^2\} \leq 0.25$. Figure (1) illustrates the range of admissible c's for $c > 0$, and compares it to Bahadur's upper bound. The inequalities (13) are tighter than (14) for $c > 0$, and for $c < 0$. But in both cases the upper bounds decrease with n and p. In Bahadur's distribution all higher-order correlation coefficients equal to zero. The fact that his distribution differs from (8) imply that some higher-order correlations in (8) differ from zero. Kaniovski (2009) shows that Bahadur's distribution is not second-order invariant.

3.1 The main result

In the following, we assume that c satisfies inequality (13) so that (8) defines a joint probability distribution. Equation (8) can be used to compute Condorcet's probability

$$\sum_{t_{\mathbf{v}}=\frac{n+1}{2}}^{n} \pi^*_{\mathbf{v}}. \tag{15}$$

[4]It remains an open question if an upper-bound common to all distributions consistent with the homogeneous jury model can be found.

The main result of this paper is as follows:

Proposition (Second-Order Invariance). *Given the solution (8) and the condition (13), the probability of a homogeneous jury collectively making the correct decision is independent of the common second-order correlation coefficient, provided the collective decision is made under simple majority rule.*

Proof. Substitute the solution (8) in the probability (15). To prove the theorem, show

$$\sum_{t_{\mathbf{v}}=\frac{n+1}{2}}^{n}\Big[0.5n(n-1)+2t_{\mathbf{v}}(t_{\mathbf{v}}-n)\Big]=0. \tag{16}$$

Recall that $t_{\mathbf{v}}=\sum_{i=1}^{n}v_i$ is the number of correct votes in the voting profile $\mathbf{v}$. To simplify the notation, I will drop the subscript on $t_{\mathbf{v}}$. To evaluate the *double* sums $\sum_{t=\frac{n+1}{2}}^{n}t$ and $\sum_{t=\frac{n+1}{2}}^{n}t^2$, use the following identities: $xC_n^x=nC_{n-1}^{x-1}$ and $x^2C_n^x=n(n-1)C_{n-2}^{x-2}+nC_{n-1}^{x-1}$. We have,

$$\sum_{t=\frac{n+1}{2}}^{n}n(n-1)=2^{n-1}n(n-1); \tag{17}$$

$$\sum_{t=\frac{n+1}{2}}^{n}t=\sum_{t=\frac{n+1}{2}}^{n}C_n^t t=n\sum_{t=\frac{n+1}{2}}^{n}C_{n-1}^{t-1}; \tag{18}$$

$$\sum_{t=\frac{n+1}{2}}^{n}t^2=\sum_{t=\frac{n+1}{2}}^{n}C_n^t t^2=n(n-1)\sum_{t=\frac{n+1}{2}}^{n}C_{n-2}^{t-2}+n\sum_{t=\frac{n+1}{2}}^{n}C_{n-1}^{t-1}. \tag{19}$$

In view of the identities (17)-(19), the equality (16) is equivalent to

$$\sum_{t=\frac{n+1}{2}}^{n}C_{n-1}^{t-1}-\sum_{t=\frac{n+1}{2}}^{n}C_{n-2}^{t-2}=2^{n-3}. \tag{20}$$

Let us prove the above identity. Expanding the above sums yields:

$$\left(C_{n-1}^{\frac{n-1}{2}}-C_{n-2}^{\frac{n-3}{2}}\right)+\left(C_{n-1}^{\frac{n+1}{2}}-C_{n-2}^{\frac{n-1}{2}}\right)+\cdots+\left(C_{n-1}^{n-1}-C_{n-2}^{n-2}\right)=C_{n-2}^{\frac{n-1}{2}}+C_{n-2}^{\frac{n+1}{2}}+\cdots+0.$$

The resulting sum equals one half of the sum $\sum_{t=0}^{n}C_{n-2}^t=2^{n-2}$, which is symmetric about the summands $C_{n-2}^{\frac{n-3}{2}}=C_{n-2}^{\frac{n-1}{2}}$, and which follows from the basic identity $\sum_{t=0}^{n}C_n^t=2^n$. $\square$

The probability of a jury being correct is independent of the correlation coefficient, provided the jury is homogeneous and the decision is made by voting under simple majority rule. In this case the expertise of a homogeneous jury cannot be impaired or improved by the independence of individual competencies. In order for a correlation to make a difference, either the jury must be heterogeneous or a supermajority must be the decision rule.

4 Numerical examples

Table 1 illustrates the effect of second-order correlation on the probability of attaining a correct decision for $n = 3$. In the case of equally probable and uncorrelated votes, all eight voting profiles are equally probable (A). Positive correlation makes broad coalitions more probable and tight coalitions less probable, while negative correlation has the opposite effect (B, C). Example D of competent and independent jurors corresponds to the assumptions of CJT. The jury's competence is reflected in a higher Condorcet's probability. Introducing positive correlation increases the probability of occurrence of all broad coalitions, including those with a high proportion of incorrect decisions (E). Within the subset of voting profiles in which the jury collectively decides correctly, these two effects offset exactly, leaving Condorcet's probability unchanged. Example F shows that invariance does not hold for heterogeneous juries. Examples D and E indicate that simple majority is essential for invariance.

Table 1: Examples of a jury of three jurors ($n = 3$)

					A	B	C	D	E	F
v_1	v_2	v_3	$t_{\mathbf{v}}$		$p = 0.5$ $c = 0$	$p = 0.5$ $c = 0.2$	$p = 0.5$ $c = -0.2$	$p = 0.75$ $c = 0$	$p = 0.75$ $c = 0.2$	$p_1 = 0.75$ $p_{2,3} = 0.6$ $c = 0.2$
1	1	1	3	$\surd$	0.125	0.2	0.05	0.422	0.478	0.325
1	1	0	2	$\surd$	0.125	0.1	0.15	0.141	0.122	0.167
1	0	1	2	$\surd$	0.125	0.1	0.15	0.141	0.122	0.167
1	0	0	1		0.125	0.1	0.15	0.047	0.028	0.090
0	1	1	2	$\surd$	0.125	0.1	0.15	0.141	0.122	0.083
0	1	0	1		0.125	0.1	0.15	0.047	0.028	0.025
0	0	1	1		0.125	0.1	0.15	0.047	0.028	0.025
0	0	0	0		0.125	0.2	0.05	0.016	0.072	0.118
Condorcet					0.5	0.5	0.5	0.844	0.844	0.743

$\surd$ indicates correct collective decisions

5 Summary and concluding remarks

I construct a distribution for which the probability of collectively making the correct decision is independent of the common correlation coefficient. The latter property, which I call second-order invariance, holds provided the jury is homogeneous, the collective decision is made by voting under simple majority rule. The defining property of a homogeneous jury is that the probability of occurrence of any voting profile depends on the total number of correct votes, but not on the identity of the jurors who cast them. The homogeneous jury model is an example of a representative agent model common to the social sciences.

This paper shows that, in the empirically relevant case of a dependence structure specified only up to second-order correlations, the validity of Condorcet's Jury Theorem cannot be generally attested because the marginal probabilities and second-order correlation coefficients do not uniquely define a joint probability distribution on the set of voting profiles, and there exists distributions for which the probability of the jury collectively making the correct decision by voting under simple majority rule is invariant to correlation. Bounds on the admissible correlation coefficient for which the second-order invariant distributions exist are provided.

The constrains on the second-order correlation coefficient implied in the homogeneous jury model are such that in a very competent jury each pair of jurors may be only weakly dependent. This is because in such a model each vote depends on every other vote. In this sense the dependency is 'global' as opposed to 'local', as in sequential voting models based on urn processes. The homogeneous jury model should thus be applied to voting in committees rather than in electorates. Nevertheless, the range of admissible values for the positive second-order correlation coefficients is sufficiently large to warrant interest in the homogeneous jury model. It is positive correlations that we typically find in voting data. Unlike sequential voting models that capture group dynamics, the homogeneous jury model applies in the baseline case of simultaneous and anonymous voting, or when the expertise of several experts whose opinions have been expressed individually is pooled into a collective judgment. This is the classic setting of Condorcet's Jury Theorem.

The proposed model has a number of related applications. There is a clear parallel between the literature on juries and the literature on the measurement of voting power. Kaniovski (2008b) explores this parallel and discusses several conceptual issues such as the merits and limitations of correlation as a model of preferences. The versatility of the model owes to the fact that once a joint probability distribution is found, the probabilities any event of interest can be computed. For example, Kaniovski (2008a) computes the probability of casting a decisive vote in simple-majority games with equal voting weights, which can be interpreted as an extended measure of voting power in the sense of Banzhaf (1965).

A Solution to the optimization problem

Define an index function of a voting profile

$$\mathcal{V}(\mathbf{v}) = 1 + \sum_{k=1}^{n} 2^{n-k} v_k. \tag{21}$$

The index function orders voting profiles in the ascending order of the decimals represented by the corresponding binary vectors. Write the Lagrangian as

$$\begin{aligned} \mathcal{L}(\mathbf{x}) &= 0.5 \sum_{\mathbf{v}\in\mathbf{V}} \Big[x_{\mathcal{V}(\mathbf{v})} - p^{\sum_{i=1}^n v_i} q^{n-\sum_{i=1}^n v_i} \Big]^2 + \lambda \Big[\sum_{\mathbf{v}\in\mathbf{V}} x_{\mathcal{V}(\mathbf{v})} - 1 \Big] + \\ &+ \sum_{i=1}^{n} \mu_i \Big[\sum_{\mathbf{v}\in\mathbf{V}(i)} x_{\mathcal{V}(\mathbf{v})} - p \Big] + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{i,j} \Big[\sum_{\mathbf{v}\in\mathbf{V}(i,j)} x_{\mathcal{V}(\mathbf{v})} - (p^2 + pqc_{i,j}) \Big]. \end{aligned} \tag{22}$$

The first order condition for every $\mathbf{v} \in \mathbf{V}$

$$x_{\mathcal{V}(\mathbf{v})} = p^{\sum_{i=1}^n v_i} q^{n-\sum_{i=1}^n v_i} - \lambda - \sum_{i=1}^{n} \mu_i v_i - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{i,j} v_i v_j. \tag{23}$$

Substitute (23) in each of the three constraints (4)-(6). To evaluate the sums, use the cardinality $|\mathbf{V}| = 2^n$, $|\mathbf{V}(i)| = 2^{n-1}$ and $|\mathbf{V}(i,j)| = 2^{n-2}$. On substitution in the first constraint

$$4\lambda + 2\sum_{i=1}^{n} \mu_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \kappa_{i,j} = 0. \tag{24}$$

Substituting (23) in the second constraint, one needs to differentiate coordinates to the left and right of the i-th coordinate:

$$4(\lambda+\mu_i)+2\left(\sum_{\substack{j=1\\j\neq i}}^{n}\mu_j+\sum_{j=1}^{i-1}\kappa_{j,i}+\sum_{j=i+1}^{n}\kappa_{i,j}\right)+\sum_{\substack{k=1\\k\neq i}}^{n-1}\sum_{\substack{l=k+1\\l\neq i}}^{n}\kappa_{k,l}=0. \tag{25}$$

which in view of (24) simplifies to

$$2\mu_i+\sum_{j=1}^{i-1}\kappa_{j,i}+\sum_{j=i+1}^{n}\kappa_{i,j}=0. \tag{26}$$

Similarly, substituting (23) in the third constraint, one must differentiate coordinates to the left of the i-th coordinate, to the right of the j-th coordinate, and in between:

$$\begin{aligned}&2^{4-n}pqc_{i,j}+4(\lambda+\mu_i+\mu_j+\kappa_{i,j})+\\&+2\left(\sum_{\substack{k=1\\k\neq i,j}}^{n}\mu_k+\sum_{\substack{k=i+1\\k\neq j}}^{n}\kappa_{i,k}+\sum_{k=1}^{i-1}\kappa_{k,i}+\sum_{l=j+1}^{n}\kappa_{j,l}+\sum_{\substack{l=1\\l\neq i}}^{j-1}\kappa_{l,j}\right)+\sum_{\substack{k=1\\k\neq i,j}}^{n-1}\sum_{\substack{l=k+1\\l\neq i,j}}^{n}\kappa_{k,l}=0.\end{aligned} \tag{27}$$

In view of (24) and (26) the above expression simplifies to

$$\kappa^*_{i,j}=-2^{4-n}pqc_{i,j}. \tag{28}$$

Plugging (28) in (24) and then in (26) yields the solution

$$\begin{aligned}\pi^*_{\mathbf{v}} &= p^{\sum_{i=1}^n v_i}q^{n-\sum_{i=1}^n v_i}+2^{2-n}pq\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}c_{i,j}-2^{3-n}pq\sum_{i=1}^{n}v_i\left(\sum_{j=1}^{i-1}c_{j,i}+\sum_{j=i+1}^{n}c_{i,j}\right)+\\&+ 2^{4-n}pq\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}c_{i,j}v_iv_j.\end{aligned} \tag{29}$$

Setting $c_{i,j}=c$ allows us to simplify the above solution to

$$\pi^*_{\mathbf{v}}=p^{\sum_{i=1}^n v_i}q^{n-\sum_{i=1}^n v_i}+2^{2-n}pqc\left(0.5n(n-1)-2(n-1)\sum_{i=1}^{n}v_i+4\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}v_iv_j\right). \tag{30}$$

It can be further simplified by letting $t_{\mathbf{v}}=\sum_{i=1}^n v_i$. Since $2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}v_iv_j=t_{\mathbf{v}}^2-t_{\mathbf{v}}$,

$$\pi^*_{\mathbf{v}}=p^{t_{\mathbf{v}}}q^{n-t_{\mathbf{v}}}+2^{2-n}pqc\Big[0.5n(n-1)+2t_{\mathbf{v}}(t_{\mathbf{v}}-n)\Big]. \tag{31}$$

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