Straffin meets Condorcet. What can a voting power theorist learn from a jury theorist?

Serguei Kaniovski*

September 5, 2007

Abstract

This paper critically examines Straffin's probabilistic characterizations of the classical measures of voting power. It then introduces an extension of the Banzhaf measure of voting power to arbitrarily distributed and correlated votes, and discusses the advantages and disadvantages of correlation as a statistical tool for modeling actual voting behavior. The paper is motivated by the literature on the Condorcet Jury Theorem, which has already probed the issue in a different setting. Numerical examples show how heterogeneity among the members of a voting body, when expressed in terms of correlation between votes, can induce actual power distributions different from those implied by equiprobable and independent votes.

JEL-Codes: C63, D72
Key Words: Banzhaf measure, Condorcet Jury Theorem, correlated votes

* *Acknowledgments:* This paper benefited from comments by Dennis Leech and three anonymous referees.
Austrian Institute of Economic Research (WIFO), P.O. Box 91; A-1103 Vienna, Austria.
Email: `serguei.kaniovski@wifo.ac.at`.

1 Introduction

The indices by both Banzhaf (1965) and Shapley and Shubik (1954) measure the distribution of *a priori* voting power that follows from the rules of a voting body alone. In a weighted voting game, the rules are given by the weights attached to the members' votes and the quota of total weights needed to pass a motion. The exclusive focus on the rules makes *a priori* power different from actual power defined by the probability of casting a critical vote.

Straffin (1977) provides a characterization of the Banzhaf measure (Bz) and the Shapley-Shubik Index in terms of stochastic models. Straffin's models assume independent votes. However, there are compelling reasons to view the votes as correlated rather than independent. While it is beyond the scope of this paper to review the literature on possible sources of dependence, three factors stand out: preferences, strategies and information. Members of a voting body are likely to have different preferences with respect to a motion. Common or opposite preferences may lead to positively or negatively correlated votes. Second, imperfectly informed members may vote strategically. Strategy, as a contingent action plan, will not be independent by definition. Third, and following Straffin, members of a voting body may follow common standards when evaluating a motion, so that the votes in favor will correlate positively. One example of a common standard is common information.

This paper focuses on the Bz (absolute) measure of power. Implicit in *a priori* Bz power is the equal treatment of all coalitions. This property cannot realistically be maintained if preferences, strategies and information are to enter the measurement of power. Taking a differentiated view of the probabilities of occurrence of coalitions seems to be a natural way of taking into account the heterogeneity among members of a voting body that goes beyond voting weights (e.g. Laruelle and Valenciano 2002).

This paper is inspired by the literature on the Condorcet Jury Theorem (CJT), which has already probed the issue. This literature is not about the voting power of an individual member, but rather about the problem-solving power of the voting body as a group. A key question in this literature concerns the effect of individual competences on collective competence when the former are not independent. Voting in juries may be different from voting in general committees insofar as, ideally, jurors should have identical preferences, namely to convict the guilty and acquit the innocent. Different individual decisions in juries will likely arise from differences in information, as opposed to heterogeneous preferences. The statistical consequences will nevertheless be the same. These statistical consequences provide a unifying theme between the literature on the Condorcet Jury Theorem and the literature on voting power. A voter is the more powerful the more frequently her vote is critical. But this will depend on circumstances created by other voters casting their votes so that she has opportunities to be critical. The likelihood of these circumstances occurring and hence the actual voting power will depend on the stochastic properties of the votes.

Correlation also seems to be a general way of taking voters' preferences into account, and the need to do so is debated in the literature on voting power (e.g. Napel and Widgrén 2004, Braham and Holler 2005). The generality derives from correlation expressing only the probabilistic tendency between a pair of members toward certain positions, rather than deterministic positions.

It is of equal importance that correlation introduces an element of realism to the analysis of *a priori* power, while preserving the basic structure of the Bz measure and everything we know about weighting voting games which follows from the properties of characteristic functions.

Section 2 discusses Straffin's theorems leading to the interpretation of the classical measure of voting power as the probability of casting a critical vote. It emphasizes the importance of stochastic independence and the paucity of Straffin's Homogeneity Assumption as a model of correlated voting. The discussion then turns to the probability of casting a critical vote as a measure of voting power, and the advantages and disadvantages of correlation as a statistical tool for modeling actual voting behavior. Section 3 reviews the extensions of the CJT to correlated votes, with an emphasis on Boland's (1989) opinion leader model, as its structure is broadly similar to that of the model discussed in this paper. Drawing on results by Kaniovski (2008), Section 4 discusses a numerical scheme for constructing a generalized Bz measure of voting power from the probabilities of each member's YES vote and the correlation coefficients between them. The method solves the aggregation problem of finding a distribution on the set of coalitions that is consistent with empirically estimable probabilities and correlations between individual votes. In Section 5 I use this method to construct numerical examples on Boland's opinion leader theme. They show how heterogeneity, when expressed in terms of correlation between votes, can induce actual power distributions different from those implied by equiprobable and independent votes. The last section offers concluding remarks.

2 Straffin's characterization

Straffin (1977) shows that the appropriate model for the Bz measure is one of equiprobable and independent votes, where equiprobability means that each member votes YES with $p = 0.5$. The equiprobability and independence of votes ensure the equiprobability of coalitions. The appropriate model for the SSI is one in which each member votes YES independently with an equal probability p, drawn from the uniform distribution on $[0, 1]$.[1] Straffin calls the two respective sets of assumptions Independence and Homogeneity.

[1]Straffin's original Independence Assumption says: "The p_i's are selected independently from the uniform distribution on $[0, 1]$". Leech (1990) shows that the requirement of a uniform distribution is unduly strong in the case of the Bz measure – it suffices that p_i's are selected independently from a distribution on $[0, 1]$, having 0.5 as its expected value. But for all practical purposes Straffin's formulation is equivalent to a Bernoulli model with the $p = 0.5$ stated above (Felsenthal and Machover (1998, Rem. 3.1.2)).

Independence seems a valid assumption in the absence of prior knowledge about future issues on the ballot and how divided the voting body will be over these issues. It is an assumption one is compelled to make at the constitutional stage of a voting body. Homogeneity alludes to the fact that decisions by voting bodies with p close to 0 or 1 would appear positively correlated to someone *who does not know the value of* p, suggesting a high degree of commonality among the members. But even then it is highly restrictive, as the implied correlation coefficient is fixed at 1/3 (Felsenthal and Machover 1998, p. 210).

Two further points are worth emphasizing. First, stochastic independence of votes is essential in both models. A mere glance at Straffin's proofs suffices to see that an increment in the multilinear extension of a simple game incurred by the addition of a member's vote can only be interpreted as the expected – in the sense of mathematical expectation – impact of this member on the outcome of voting if all votes are stochastically independent. Second, the two models cannot be directly compared because they have different spaces of elementary outcomes and hence also different probability spaces.

2.1 The Banzhaf measure

Let the set of voters be $N = \{1, 2, \ldots, n\}$. A coalition, that is a set of YES voters, is defined by any subset of N, $S \subseteq N$. There will be 2^n such coalitions. The characteristic function is the map w from the set of all coalitions to $\{0, 1\}$ such that $wS = 1$ if S is winning, and $wS = 0$ otherwise. If the i-th vote is critical for S, then $wS - w(S\backslash\{i\}) = 1 - 0 = 1$. If the i-th vote is not critical for S, then $wS - w(S\backslash\{i\}) = 0 - 0 = 0$, or $wS - w(S\backslash\{i\}) = 1 - 1 = 0$. In the first case, the i-th vote is insufficient for S to win; in the second case, it is unnecessary for S to win.

The Bz measure of voting power can be defined as

$$\beta_i = \frac{1}{2^{n-1}} \sum_{S \subseteq N, i \in S} (wS - w(S \backslash \{i\})). \tag{1}$$

The sum equals the number of coalitions in which i is critical, called the Bz score of i.

In a binary voting game $P(i \text{ votes YES}) + P(i \text{ votes NO}) = 1$, so that

$$P(i \text{ is critical}) = P(i \text{ is critical} \cap i \text{ votes YES}) + P(i \text{ is critical} \cap i \text{ votes NO}). \tag{2}$$

In terms of conditional probabilities,

$$\begin{aligned} P(i \text{ is critical}) &= P(i \text{ is critical} \mid i \text{ votes YES}) P(i \text{ votes YES}) + \\ &+ P(i \text{ is critical} \mid i \text{ votes NO}) P(i \text{ votes NO}). \end{aligned} \tag{3}$$

If the votes are equiprobable and independent, then all 2^{n-1} coalitions in which i votes YES would occur with probability 0.5^{n-1}, and β_i would be the probability of i being critical by voting YES. Similarly, β_i would be the probability of i being critical by voting NO.

In the general case of arbitrarily distributed and possibly dependent votes, the two conditional probabilities in (3) will be different, so that the actual power of a member will depend on whether she votes YES or NO.

Remark 1. *In the general case of arbitrarily distributed and possibly dependent votes, the probability of i being critical by voting YES will be*

$$P(i \text{ is critical} \mid i \text{ votes YES}) = \frac{1}{p_i} \sum_{S \subseteq N, i \in S} \pi_S (wS - w(S \backslash \{i\})).$$

Here $p_i \in [0, 1]$ is the probability that i votes YES, and $\pi_S \in [0, 1]$ the probability of occurrence of coalition S given by a joint probability distribution π on the set of all coalitions. Two questions suggest themselves: which distributions occur empirically and which can be derived from plausible stochastic assumptions on votes. The former question has been addressed before. For example, Heard and Swartz (1998) estimate π for coalitions in the Supreme Court of Canada using Bayesian techniques and compute the above measure, which they call the empirical Banzhaf index. Finding an answer to the second question requires solving an aggregation problem. Section 4 discusses a numerical scheme for constructing π for given probabilities and correlation coefficients between the votes.

Remark 2. *The characteristic function w does not depend on the distribution π.*

Coalitions that are winning under equally probable and independent votes continue to do so when the votes lose either property; what changes are the probabilities of their occurrence. Assigning different weights to members of a given voting body, or changing the quota, may change the characteristic function of the game, but not the probability distribution. This fact is important because it means that the properties of the characteristic function can be studied separately from those of the probability distribution. It makes symmetric simple-majority games an interesting subject for analysis, as the effect of the characteristic function on power in those games is trivial, whereas the effect of the probability distribution is not.

2.2 Probability as a measure of power

The probability of being critical describes the ability to change the outcome of voting. As such it is a measure of the actual voting power that fits well with the Weberian notion that "power is the probability that one actor within a social relationship will be in position to carry out his own will despite resistance, regardless of the basis on which this probability rests" (Martin 1971, p. 241).[2] It is a compound measure which takes into account both the calculus of coalitions that follows from the rules of the voting body and the likelihood of their occurrence.

Power is an elusive concept, and probability is not a flawless measure of power. If the rules of a voting body are such that a member's *a priori power* is zero (the member is a dummy), then nothing except a change of the rules could change this fact. However, it is possible for a member to be powerless despite her having some *a priori* power. This would be the case if all coalitions in which the member is critical have zero probabilities of occurrence. If coalitions in which a member is critical never form due to other members' actions, then that member would be powerless in the sense of not being able to realize her own will in a collective action. It should be noted that the probability of being critical is not at odds with Braham and Holler's (2005) critique of preference-based measures of power. Probability still describes only a potential for being critical. But it is a potential derived from circumstances which are in part outside of the framework of simple voting games, such as preferences, strategies and information.

2.3 Correlations as preferences

Correlation between votes offers a flexible way of modeling heterogeneity among the members of a voting body. Take the example of heterogeneous preferences. Two common approaches are used to represent members' preferences: a spatial one and one involving a direct restriction on the set of coalitions. Preferences in a spatial representation are points in a suitably labeled Euclidean space, such as a policy space spanned by the quantities of two public goods. The degree of commonality between preferences can be expressed by the distance between the members' preferred (ideal, or bliss) points. Spatial representations often are deterministic and therefore

[2] For a discussion and a critique of the Weberian notion of power see Morriss (2002).

rigid, while correlations only express the probabilistic tendency of members toward certain outcomes. While correlations can be easily estimated from ballot data, it is difficult to choose ideal points for members of a real voting body. Explicit restrictions on the set of possible coalitions are equally rigid, as is demonstrated by the paradox of quarreling members. If two members have a quarrel, then any coalition containing both members is impossible and will not count toward their Bz scores (Brams 1975).

Correlation seems to be a flexible and empirically promising way of modeling preferences, but this method is not without limitations. First, correlation is a pairwise property of random variables. It cannot capture relationships between an individual member and a bloc of members. This entails no loss of generality if voting blocs are deterministic, in the sense that each insider votes in unison with all other insiders with probability one, because then pairwise correlation between an outsider and a bloc is equivalent to pairwise correlation between the outsider and a hypothetical member holding the total weight of the bloc in votes. The literature on voting power typically studies deterministic blocs.

Second, while in general three random variables can be negatively correlated pairwise, negative correlations present an awkward constraint in binary choice situations. If one member votes YES whenever another votes NO, and vice versa, then by virtue of binary choice a third member's vote cannot be simultaneously discordant with the former two. Three members cannot be mutually contrarian. Positive correlations do not impose such a constraint, as they reflect common rather than contrarian tendencies. Positive correlations between the YES votes make all such votes more probable than they would otherwise be if the votes were independent. Most of the literature discussed in the next section assumes positive correlations.

Third, the property of being positively correlated is not transitive, and so cannot be used to define a transitive preference relationship. Langford, Schwertman and Owens (2001) show that from the correlation coefficients satisfying $c_{x,y} > 0$ and $c_{y,z} > 0$ it does not follow that $c_{x,z} > 0$, unless $c_{x,y}^2 + c_{y,z}^2 > 1$. The property of being positively correlated is only transitive when correlation between any two of the three random variables is sufficiently high. For example, $c_{x,y} = 0.75$ and $c_{y,z} = 0.70$ satisfies this inequality, but $c_{x,y} = 0.75$ and $c_{y,z} = 0.60$ does not.

3 Condorcet's Jury Theorem (CJT)

In his famous 1785 essay, Condorcet rationalized entrusting important decisions to a collective rather than an individual, and found the simple majority to be the best mode for a collective to reach its verdict. CJT is a mathematical formalization of his ideas (Young 1988). In its classic version the theorem rests on five assumptions: 1) the jury must choose between two alternatives (of which one is referred to as correct); 2) the jury makes its decision by voting under the simple majority rule; 3) each juror is competent (has more than an even probability of voting for the correct alternative); 4) all jurors are equally competent (have equal probabilities); and 5) each juror makes his or her decision independently of all other jurors. For a jury that satisfies all the above criteria, the following is known to be true:

1. any jury comprising an odd number greater than one of jurors is more likely to select the correct alternative than any single juror and

2. this likelihood tends to a certainty as the number of jurors tends to infinity.

It is customary to refer to 1 and 2 as the non-asymptotic and asymptotic parts of the theorem. In the literature on CJT, a jury that violates the assumption of equal competence is called *heterogeneous*, as opposed to *homogeneous*.

Remark 3. *A homogeneous jury is a Homogeneous voting body in the sense of Straffin (1977).*

I will return to this point in Section 5. For the moment, note that the above remark holds for both competent and incompetent juries as long as they are homogeneous. Similarly, Sraffin's Independence Assumption coincides with that of a *randomizing* jury in which each juror is correct with probability 0.5.

A large body of research focuses on the consequences of relaxing the assumptions of equal competences (homogeneity) and independence for the validity of the theorem. Both assumptions are unrealistic, as jurors are subject to many individual and contextual factors that are likely to influence their decisions in different ways. These factors include differences or similarities in education, experience or ideologies, the presence of common and asymmetric information, and the presence of strategic behavior. Realistically, the jurors should have different but overlapping competences, with sets in a Venn diagram providing a useful analogy here. The competences of different jurors could overlap in different ways, for example: Joe and Ann went to the same college, Joe and Peter support the same political party, Ann and Peter bowl together on weekends, and all of them face the same evidence in court.

The assumption of competence is crucial to the asymptotic part of the theorem as the converse often leads to an opposite result: As the number of incompetent jurors tends to infinity, the jury arrives at the incorrect decision with probability one. Berend and Paroush (1998) derive a necessary and sufficient condition required for the asymptotic part of the theorem to hold. As a condition on the mean of jurors' competences, it encompasses the case of homogeneous juries. Independence is still required.

The non-asymptotic part of the theorem does not generally hold for heterogeneous juries. In a heterogeneous jury all jurors are competent, but in different ways. Ben-Yashar and Paroush (2000) construct an example of a jury consisting of one juror with higher competence and two jurors with lower competence, in which the likelihood of the jury choosing the correct alternative using the simple majority rule is lower than the likelihood that the most competent juror chooses it on her own (see also Nitzan and Paroush 1984, Berend and Sapir 2005). Ben-Yashar and Paroush (2000) propose a refinement of the CJT that remains valid regardless of the specific distribution of competence within a jury. They show the non-asymptotic part of CJT to hold if the average competence of a randomly chosen group of more than three jurors is higher than that of any single juror.

The validity of CJT when a supermajority instead of simple majority is required for the jury to reach a decision has been discussed in Nitzan and Paroush (1984), Ben-Yashar and Paroush (2000), and Kanazawa (1998). Fey (2003) shows both parts of the theorem to hold in a sufficiently large jury "as long as the average competence of the voters is greater than the fraction of votes needed for passage" (p. 28). Hence, the theorem also holds for heterogeneous juries using supermajority rules, although independence remains essential.

Competence seems to be a natural assumption for a group of experts. The same cannot be said of stochastic independence.[3] There are at least three extensions of CJT to correlated votes. These include the opinion leader model by Boland (1989), the model based on an urn process by Berg (1985), and a general framework of conditional probabilities by Ladha (1992). Boland studies a heterogeneous jury consisting of an opinion leader and several followers. Berg and Ladha study *ex ante* homogeneous juries in which each vote has an equal probability of being correct and each pair of votes is correlated with the same coefficient of correlation. Although each model involves a different conceptual extension of the probabilistic setting of the theorem, all three conclude that the theorem remains valid only if the correlations between the votes do not exceed certain thresholds.

[3]The fact that the jury is confronted with the *same* evidence already suggest probabilistic dependence of a juror's vote on the quality of evidence (Dietrich and List 2004). A popular book by James Surowiecki contains many fascinating examples of a phenomenon commonly known as "group think" (Surowiecki 2005).

3.1 Boland's opinion leader model

Consider a voting body comprised of two members, i and j. Suppose i votes YES with probability p_i, j votes YES with probability p_j, and their votes correlate with a coefficient of correlation $c_{i,j}$. Define the probabilities of the four possible voting profiles as: $P\{v_i = 1, v_j = 1\} = \pi_1$, $P\{v_i = 1, v_j = 0\} = \pi_2$, $P\{v_i = 0, v_j = 1\} = \pi_3$, $P\{v_i = 0, v_j = 0\} = \pi_4$, where 1 and 0 respectively indicate the YES and NO vote. We have: $\pi_1 + \pi_2 = p_i$, $\pi_1 + \pi_3 = p_j$ and $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$. As the covariance $cov(v_i, v_j)$ between the two Bernoulli random variables v_i and v_j, such that $E(v_i) = p_i$ and $E(v_j) = p_j$, is $E(v_i v_j) - E(v_i)E(v_j) = \pi_1 - p_i p_j$, the coefficient of correlation $c_{i,j} = cov(v_i, v_j)/\sqrt{var(v_i)var(v_j)}$ must satisfy

$$\pi_1 = p_i p_j + c_{i,j}\sqrt{p_i(1 - p_i)p_j(1 - p_j)}. \tag{4}$$

Note that two uncorrelated Bernoulli random variables are independent.

Marginal probabilities p_i and p_j can be interpreted as *ex ante* probabilities of i and j voting YES. I attach the following empirical meaning to this statement: Should i alone decide on a series of motions, then the relative frequency of accepted motions would approach p_i as their number approaches infinity. Should i and j decide together, then the frequency of unanimously accepted motions would approach π_1.

The above formulation does not assume anything about the dynamics of the voting procedure or the nature of probabilistic dependence, but such interpretations can be easily introduced by rewriting (4) as a conditional probability $\pi_1 = p_i + \frac{c_{i,j}}{p_j}\sqrt{p_i(1 - p_i)p_j(1 - p_j)}$. Events v_i and v_j can be interpreted in several ways. Boland (1989) assumes sequential voting in which j's vote is followed by i's, who observes it prior to voting.[4] If j votes YES and $c_{i,j}$ is positive, then

[4]In a different version of the model, Boland, Proschan and Tong (1989) view j as an exogenous factor, such as

chances are that i will also vote YES. Thus, there is a state dependence on how j has voted. Boland's model involves a jury consisting of an opinion leader whose vote j is independent, and an even number of followers whose votes correlate with that of j. The votes of the followers are conditionally independent, given the opinion leader's vote. If p is the probability that a juror will make the correct decision, $q = 1-p$, and c is the correlation coefficient between the vote of a follower and the leader's vote, then we have $P\{v_i = 1|v_j = 1\} = p+cq$, $P\{v_i = 0|v_j = 1\} = q-cq$, $P\{v_i = 1|v_j = 0\} = p - cp$, $P\{v_i = 0|v_j = 0\} = q + cp$. This follows from the definition of conditional probability and the following simple Lemma:

Lemma 1. *Let v_i and v_j be two Bernoulli random variables with $E(v_i) = p_i$ and $E(v_j) = p_j$. Further, let*

$$P\{v_i = 1, v_j = 1\} = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \quad (5)$$

$$P\{v_i = 0, v_j = 1\} = (1-p_i)p_j + \bar{c}_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \quad (6)$$

$$P\{v_i = 1, v_j = 0\} = p_i(1-p_j) + \hat{c}_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}; \quad (7)$$

$$P\{v_i = 0, v_j = 0\} = (1-p_i)(1-p_j) + \tilde{c}_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}. \quad (8)$$

Then, $c_{i,j} = -\bar{c}_{i,j}$, $c_{i,j} = -\hat{c}_{i,j}$, $c_{i,j} = \tilde{c}_{i,j}$.

Boland's model describes a situation in which the opinion leader's vote is followed by the simultaneous voting of his followers, whereby the nature of events involved becomes sequential. The above is not the only way to model probabilistic dependence between votes. Several alternatives have been proposed in the literature, including the urn model by Berg (1985) in the context of voting power, and Berg (1993) in the context of jury theorems. Berg's approach is

the quality of evidence presented in court. See Dietrich and List (2004) for a related approach based on conditional probabilities of the jury being correct given that the evidence can be misleading.

based on an urn process. The sequential nature of events follows by construction of an urn scheme, in which colored balls are drawn one at a time and then replaced by one or several balls of the same color. A model based on an urn process implicitly assumes that the *ex post* probability of a vote being YES may change every time a vote is cast. The voting procedure becomes dynamic because each vote alters the composition of the urn, thereby creating dependence between votes. Berg concludes that positive correlation decreases the probability of the jury being correct, while negative correlation, representing contrarian or rivaling competences, has the opposite effect. Both models imply a state dependence in the process of reaching a decision, with the possibility of a lock-in on the incorrect alternative (Page 2006).

A sequential model is not universally applicable. In a voting by roll-call situation, one could condition the probability of each vote turning out in favor of an alternative on all preceding votes, as these are observed prior to the vote being cast. An urn model is appropriate in this situation and, in general, for modeling group dynamics among members of a voting body. However, an urn model is less appropriate in the very common case of simultaneous and anonymous voting, or when the expertise of several experts whose opinions have been expressed individually is pooled into a collective judgment.

Traditionally, the literature on jury theorems and voting power assumes sincere voting. A vote is sincere if it truthfully reflects the voter's preferences. Studies on the effect of informational asymmetries on voting behavior in juries show that sincere voting is not rational and cannot be an equilibrium behavior in general (Feddersen and Pesendorfer 1996). Boland's approach and its extension in this paper are compatible with both sincere and strategic behavior, as well as many other types. However, the key issue is not the rationale behind the individual behavior but the statistical consequences of this behavior.

Gelman, Katz and Tuerlinckx (2002) propose another approach that accommodates correlated votes based on the Ising model from statistical mechanics. In their model votes are equiprobable but correlated. Correlations are not explicitly defined, but follow implicitly from a parameter of spatial proximity. They derive a specification for the variance of proportional vote differentials between constituencies of different sizes and test it, as well as Penrose's (1946) square-root rule of equal representation in two-tier voting systems which follows from the Bernoulli model in Straffin's Independence Assumption using the data from U.S. Presidential elections. Both specifications find little empirical support (see also Gelman, Katz and Boscardin 2003, Gelman, Katz and Bafumi 2004).

4 Computing the Bz measure in the general case

In order to introduce varying probabilities and correlations between votes into the generalized Bz measure in Remark 1, we must construct a probability distribution on the set of coalitions that satisfies given probabilities and correlation coefficients. For $n = 2$ this is easily accomplished by simultaneously solving the following four equations $\pi_1+\pi_2 = p_i$, $\pi_1+\pi_3 = p_j$, $\pi_1+\pi_2+\pi_3+\pi_4 = 1$, $\pi_1 = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)}$, subject to $\pi_1, \pi_2, \pi_3, \pi_4 \in [0, 1]$.

To formulate an analogous system of equations for an arbitrary number of voters I will use the following notation. A voting profile is given by a binary vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, whose i-th coordinate $v_i = 1$ if i votes YES, and $v_i = 0$ if i votes NO. Let $\mathbf{V}$ be the set of all voting profiles, $\mathbf{V}(i)$ the set of voting profiles in which member i votes YES, that is, the set of all binary vectors $\mathbf{v}$ such that $v_i = 1$, and $\mathbf{V}(i,j) = \mathbf{V}(i) \cap \mathbf{V}(j)$ the set of voting profiles in which members i and j both vote YES, that is, the set of all binary vectors $\mathbf{v}$ such that $v_i = v_j = 1$. Sets $\mathbf{V}$, $\mathbf{V}(i)$ and $\mathbf{V}(i,j)$ respectively contain 2^n, 2^{n-1} and 2^{n-2} elements. For example, for $n = 3$ there will be eight voting profiles 1:(1,1,1), 2:(1,1,0), 3:(1,0,1), 4:(1,0,0), 5:(0,1,1), 6:(0,1,0), 7:(0,0,1), and 8:(0,0,0). The set $\mathbf{V}$ contains all eight vectors. The set $\mathbf{V}(2)$ contains the four vectors 1, 2, 5 and 6, as only they have 1 in the second coordinate. The set $\mathbf{V}(2,3)$ contains two vectors, 1 and 5, as only they have 1 in the second and third coordinates.

Using the above notation, the system for n voters can be written as

$$\pi_{\mathbf{v}} \in [0,1], \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}; \tag{9}$$

$$\sum_{\mathbf{v}\in\mathbf{V}} \pi_{\mathbf{v}} = 1, \quad \text{for all} \quad i = 1,2,\ldots,n; \tag{10}$$

$$\sum_{\mathbf{v}\in\mathbf{V}(i)} \pi_{\mathbf{v}} = p_i, \quad \text{for all} \quad i = 1,2,\ldots,n; \tag{11}$$

$$\sum_{\mathbf{v}\in\mathbf{V}(i,j)} \pi_{\mathbf{v}} = p_i p_j + c_{i,j}\sqrt{p_i(1-p_i)p_j(1-p_j)} \quad \text{for all} \quad 1 \le i < j \le n. \tag{12}$$

Constraints (9)-(12) define a convex subset (polytope) in $\mathbb{R}^{2^n}$, $C \subset \mathbb{R}^{2^n}$. Any point belonging to C is a suitable probability distribution. Such a distribution exists whenever $C \neq \varnothing$. For $n \geq 3$ the above system will typically have infinitely many solutions, as with $n \geq 3$ probabilities and $\binom{n}{2}$ coefficients of correlation it would comprise $1 + n + \binom{n}{2}$ equations with 2^n unknowns. For a particular solution, Kaniovski and Pflug (2007) propose choosing the one which is closest in the sense of least squares to the probability distribution in the case of independent votes. This solution can be obtained by solving the following quadratic optimization problem

$$\min_{\pi_{\mathbf{v}}} \quad \frac{1}{2}\sum_{\pi_{\mathbf{v}}}\left[\pi_{\mathbf{v}} - \prod_{i=1}^{n} p_i^{v_i}(1-p_i)^{(1-v_i)}\right]^2, \tag{13}$$

subject to constraints (9)-(12). Clearly, the inputs must satisfy $p_i \in [0,1]$ and $c_{i,j} \in [-1,1]$, and the correlation matrix collated from $c_{i,j}$'s must be positive semi-definite.

The strict convexity of the objective function ensures a unique solution. In principle, any probability vector of length 2^n can be used as a criterion vector for computing the smallest sum of squared deviations. The chosen vector corresponds to the probability distribution in the case of independent votes, which is a natural benchmark.

Solving the most general optimization problem with arbitrary probabilities and correlation coefficients requires numerical optimization. Kaniovski (2008) analytically solves a slightly less general problem, in which all the probabilities are identical but the correlation coefficients may vary. An analytical solution is obtained by the method of Lagrangian multipliers (Appendix).

Proposition 1. *Let $p_i = p \in [0,1]$ for all $i = 1, 2, \dots, n$ be the probability of i-th member voting YES and $c_{i,j} \in [-1,1]$, $1 \le i < j \le n$, the correlation coefficient between any two such votes. Setting $q = 1 - p$, the probability of occurrence of a voting profile is given by*

$$\begin{aligned} \pi^*_{\boldsymbol{v}} &= p^{\sum_{i=1}^{n} v_i} q^{n-\sum_{i=1}^{n} v_i} + 2^{2-n} pq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{i,j} - 2^{3-n} pq \sum_{i=1}^{n} v_i \left(\sum_{j=1}^{i-1} c_{j,i} + \sum_{j=i+1}^{n} c_{i,j} \right) + \\ &+ 2^{4-n} pq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{i,j} v_i v_j, \end{aligned} \tag{14}$$

provided $\pi^*_{\boldsymbol{v}} \ge 0$.

For a homogeneous jury, in which $c_{i,j} = c$ for all $1 \le i < j \le n$, the solution simplifies to

$$\pi^*_{\mathbf{v}} = p^{\sum_{i=1}^{n} v_i} q^{n-\sum_{i=1}^{n} v_i} + 2^{2-n} pqc \Bigg(\frac{n(n-1)}{2} - 2(n-1) \sum_{i=1}^{n} v_i + 4 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} v_i v_j \Bigg). \tag{15}$$

The model is completely general in that it admits varying probabilities and correlation coefficients. If distributions that satisfy the inputs indeed exist, the numerical scheme will find one such distribution. Since the analytical solution does not take into account the complementary slackness conditions (9), it yields the same distribution only if all voting profiles have positive probability, which restricts the admissible correlation coefficients. Applied work should use numerical optimization with the full set of constraints imposed.

Equation (14) can be used to compute the probability of occurrence of any of the 2^n voting profiles. Once a distribution is found, one can compute the Bz measure in *any weighted voting game, but doing so still involves finding the winning coalitions in which the member is critical.* The problem of finding such coalitions cannot, in general, be solved analytically. Moreover, finding such coalitions poses a separate problem, as the characteristic function is independent of the stochastic properties of the votes (Remark 2). Kaniovski (2008) provides a closed-form expression for the Bz measure in a symmetric simple-majority game, in which each vote has an equal probability of being YES, but not necessarily 0.5, and each pair of votes is correlated with the same coefficient of correlation.

5 Examples

This section presents twelve numerical examples that extend Boland's opinion leader model described in Section 3.1 to a richer and more realistic setting. Consider the simplest case of a symmetric (unweighted) simple-majority game with three members. Member L (leader) acts as an opinion leader by voting first, or by credibly announcing her vote prior to other member's voting. The other two members can be of type N, F and C. Type N is neutral to L, and their votes are independent. Type F (follower) tends to agree with L on average, and their votes are positively correlated. Type C (contrarian) tends to disagree with L on average, and their votes are negatively correlated. Type N is also neutral to F and C, while F and C are obviously contrarian. To fix the parameters in terms of probabilities and correlation coefficients, let the *ex ante* probability of voting YES be $p = 0.5$ for all members, and the correlation matrix be

	L	N	F	C
L	1	0	0.25	−0.25
N	0	1	0	0
F	0.25	0	1	−0.25
C	−0.25	0	−0.25	1

Table 1 illustrates the influence of the constitution of the voting body on the voting powers of its members, and the effect of L's opinion leadership. The first column contains the probabilities of occurrence of eight coalitions π. The second and third columns contain the conditional probabilities of the possible coalitions when L acts as an opinion leader by voting YES or NO, denoted, respectively, as π_Y and π_N. The conditional probabilities are obtained by dividing the corresponding unconditional probabilities in the first column by the probability of L voting YES or NO (here 0.5), and setting the probability of those coalitions in which L would vote otherwise to zero. Measures BzY and BzN of a member are computed as the sum of the probabilities of occurrence of all coalitions in which the member is critical using the conditional distributions in columns two and three. For a member in position 1,2, and 3 (L, N, N in Case 1) these coalitions are, respectively, 2,3,6,7 – 2,4,5,7 – 3,4,5,6.

In Cases 1,2, and 3, L faces two voters of the same type. By assumption, votes of the same type are perfectly positively correlated, as they effectively form a deterministic voting bloc that commands a simple majority of votes, denying L any voting power.

The remaining two members in Case 1 are independent. This is the Bz ideal case described in Straffin's Independence Assumption, in which each member's actual voting power equals her *a priori* power (Section 2). As expected, the voting power of N depends neither on how she votes, nor on the behavior of L.

Substitute two neutral members by two followers F (Case 2). F's votes correlate positively with L's. Positive correlation makes broad coalitions more probable, and tight coalitions less probable. Broad coalitions represent voting profiles with a high degree of consensus or unanimity, as in the case of a grand coalition, and are characterized by a high percentage of 1's or 0's in the binary vector. Positive correlation between followers decreases their voting power from 0.5 to 0.375. In a symmetric simple majority game with an odd number of members a member is critical if she breaks a tie, and positive correlation decreases the likelihood of ties. Negative correlation makes tight coalitions more probable, and broad coalitions less probable. Case 3 portrays a situation in which two C-type members act together against L. Their voting power is higher than that of N in Case 1.

Voting bodies in Cases 4, 5, and 6 comprise members of three types. Now the voting power of L depends on whether she acts as an opinion leader. Cases 4 and 5 show that the voting power of a neutral member depends on correlations between other members' votes. In Case 4, N faces two members (L and F) who tend to take similar positions, while in Case 5 the two members (L and C) follow contrarian strategies. The voting power of the neutral member is high in the former situation and low in the latter. Case 6 is interesting because it is the only one in which the power of L increases as a result of two other members being contrarian.

In all cases in Table (1), the power of L is the same regardless of L's vote. This will not be true in general. Table 2 revisits the same six cases, but now the *ex ante* probability of voting YES is $p = 0.75$ instead of $p = 0.5$. Now the voting body is biased toward acceptance. The abundance of YES votes makes NO votes more powerful, as they are more likely to effectuate a tie. The probability of ties that can be resolved with a YES vote will be lower than the probability of ties that can be resolved with a NO vote because the former type of coalition contains more of the kind of vote that is more likely to occur. This is best seen on the example of the neutral member N.

In Section 3 we saw that a setting in which $p > 0.5$ has a natural interpretation in the literature on CJT. A jury of independent jurors in which the probability of a juror convicting the guilty is higher than 0.5 is called competent. Case 7 shows a jury of competent but not independent jurors. A comparison of distributions between Case 1 and Case 7 reveals the jury's competence reflected in higher probabilities of voting profiles in which the correct decisions are frequent (1's), and a higher probability of collectively making the correct decision.

6 Summary and conclusions

What can a power theorist learn from a jury theorist? I think a lot, if one considers that both strive for more realistic models of voting bodies. The gist of the paper and the questions it leaves open can be summarized as follows:

1. Staffin's characterization of the classical measure of voting power by Banzhaf and Shapley-Shubik unrealistically requires stochastic independence between votes. Even if accepted as a model of correlated votes, Straffin's Homogeneity Assumption is very rigid, as the implied coefficients of correlation can only assume a fixed positive value. In this paper I argue that working directly with correlated votes is a more general way of modeling truly heterogeneous voting bodies, which I demonstrate using the Banzhaf measure of voting power;

2. In doing so, I turn to the literature on the Condorcet Jury Theorem. This literature is concerned with the probability of a jury collectively making correct decisions. This

probability depends *inter alia* on correlations between the votes. Boland's opinion leader model is of particular interest as its structure is broadly similar to that of the model presented in this paper;

3. I discuss a simple generalization of the Bz measure. The measure admits any probability distribution on the set of coalitions. Further work is required in order to identify families of distributions resembling those which occur empirically, as well as those which consistently arise from different (suitably stylized) modes of individual voting behavior. The paper discussed a method that solves the problem of finding a distribution on the set of coalitions that is consistent with given probabilities and correlations between individual votes;

4. Introducing correlation between the votes appears to be a general way of capturing heterogeneity among members of a voting body due to preference, strategies, or informational asymmetries. The generality derives from a correlation expressing only the probabilistic tendency of a member toward certain positions, rather than a deterministic position. Correlation introduces an element of realism to the analysis of *a priori* power, while preserving the basic structure of the Bz measure and everything we know of weighting voting games. The fact that correlations can readily be estimated from ballot data opens up the possibility of calibrating an accurate empirical model of a heterogeneous voting body or estimating such a model from ballot data. A critical examination of pairwise correlation as a statistical tool for modeling heterogeneous preferences is provided;

5. I discuss a numerical scheme for constructing probability distributions on the set of coalitions from probabilities and correlation coefficients. The numerical scheme is general in admitting varying probabilities and correlation coefficients, with an analytical solution provided for a voting body in which the former are identical while the latter can vary;

6. Numerical examples indicate that *a priori* voting power is likely to be very different from actual power defined as the probability of casting a critical vote. The actual voting power of a member will depend on whether she votes YES or NO, and whether her vote is known to other members of the voting body prior to their voting.

A Solution to the optimization problem

Write the Lagrangian $\mathcal{L}(\mathbf{x})$ as

$$\begin{aligned}\mathcal{L}(\mathbf{x}) &= \frac{1}{2}\sum_{\mathbf{v}\in\mathbf{V}}\left[x_{V(\mathbf{v})} - p^{\sum_{i=1}^{n} v_i} q^{n-\sum_{i=1}^{n} v_i}\right]^2 + \lambda\left[\sum_{\mathbf{v}\in\mathbf{V}} x_{V(\mathbf{v})} - 1\right] + \\ &+ \sum_{i=1}^{n}\mu_i\left[\sum_{\mathbf{v}\in\mathbf{V}(i)} x_{V(\mathbf{v})} - p\right] + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\kappa_{i,j}\left[\sum_{\mathbf{v}\in\mathbf{V}(i,j)} x_{V(\mathbf{v})} - (p^2 + pqc_{i,j})\right]. \end{aligned} \tag{16}$$

The subscript $V(\mathbf{v}) = \sum_{i=1}^{n} 2^{n-i}(1 - v_i) + 1$ indicates the coordinate of $\mathbf{x}$ that corresponds to the probability of the voting outcome $\mathbf{v}$. The coordinates of $\mathbf{x}$ are indexed in the descending order of the decimals represented by the corresponding binary vectors of voting outcomes.

First-order conditions for a maximum with respect to $\mathbf{x}$ are

$$x_{V(\mathbf{v})} = p^{\sum_{i=1}^{n} v_i} q^{n-\sum_{i=1}^{n} v_i} - \lambda - \sum_{i=1}^{n}\mu_i v_i - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\kappa_{i,j} v_i v_j. \tag{17}$$

Substitute (17) in each of the three constraints (10)-(12). Use the fact that $\mathbf{V}$, $\mathbf{V}(i)$ and $\mathbf{V}(i,j)$ respectively contain 2^n, 2^{n-1} and 2^{n-2} elements. Substituting in (10)

$$4\lambda + 2\sum_{i=1}^{n}\mu_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\kappa_{i,j} = 0. \tag{18}$$

Substituting (17) in (11), differentiate coordinates to the left and the right of the i-th coordinate

$$4(\lambda + \mu_i) + 2\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \mu_j + \sum_{j=1}^{i-1} \kappa_{j,i} + \sum_{j=i+1}^{n} \kappa_{i,j}\right) + \sum_{\substack{k=1\\ k\neq i}}^{n-1} \sum_{\substack{l=k+1\\ l\neq i}}^{n} \kappa_{k,l} = 0. \tag{19}$$

which in view of (18) simplifies to

$$2\mu_i + \sum_{j=1}^{i-1} \kappa_{j,i} + \sum_{j=i+1}^{n} \kappa_{i,j} = 0. \tag{20}$$

Similarly, substituting (17) in (12), differentiate coordinates to the left of the i-th coordinate, to the right of the j-th coordinate, and inbetween

$$\begin{aligned} &2^{4-n}pqc_{i,j} + 4(\lambda + \mu_i + \mu_j + \kappa_{i,j}) + \\ &+2\left(\sum_{\substack{k=1\\ k\neq i,j}}^{n} \mu_k + \sum_{\substack{k=i+1\\ k\neq j}}^{n} \kappa_{i,k} + \sum_{k=1}^{i-1} \kappa_{k,i} + \sum_{l=j+1}^{n} \kappa_{j,l} + \sum_{\substack{l=1\\ l\neq i}}^{j-1} \kappa_{l,j}\right) + \sum_{\substack{k=1\\ k\neq i,j}}^{n-1} \sum_{\substack{l=k+1\\ l\neq i,j}}^{n} \kappa_{k,l} = 0. \end{aligned} \tag{21}$$

In view of (18) and (20) the above expression simplifies to

$$\kappa^*_{i,j} = -2^{4-n}pqc_{i,j}. \tag{22}$$

Plugging (22) in (18) and then in (20) yields the sought solution.

References

Banzhaf, J. F.: 1965, Weighted voting does not work: a mathematical analysis, *Rutgers Law Review* **19**, 317–343.

Ben-Yashar, R. and Paroush, J.: 2000, A nonasymptotic Condorcet jury theorem, *Social Choice and Welfare* **17**, 189–199.

Berend, D. and Paroush, J.: 1998, When is Condorcet's jury theorem valid?, *Social Choice and Welfare* **15**, 481–488.

Berend, D. and Sapir, L.: 2005, Monotonicity in Condorcet jury theorem, *Social Choice and Welfare* **24**, 83–92.

Berg, S.: 1985, Paradox of voting under an urn-model: the effect of homogeneity, *Public Choice* **47**, 377–387.

Berg, S.: 1993, Condorcet's jury theorem, dependency among jurors, *Social Choice and Welfare* **10**, 87–95.

Boland, P. J.: 1989, Majority systems and the Condorcet jury theorem, *The Statistician* **38**, 181–189.

Boland, P. J., Proschan, F. and Tong, Y. L.: 1989, Modelling dependence in simple and indirect majority systems, *Journal of Applied Probability* **26**, 81–88.

Braham, M. and Holler, M.: 2005, The impossibility of a preference-based power index, *Journal of Theoretical Politics* **17**, 137–157.

Brams, S. J.: 1975, *Game theory and politics*, Free Press.

Dietrich, F. and List, C.: 2004, A model of jury decisions where all jurors have the same evidence, *Synthese* **142**, 175–202.

Feddersen, T. and Pesendorfer, W.: 1996, The swing voters curse, *American Political Science Review* **86**, 408–424.

Felsenthal, D. S. and Machover, M.: 1998, *The measurement of voting power: theory and practice, problems and paradoxes*, Edward Elgar.

Fey, M.: 2003, A note on the Condorcet jury theorem with supermajority voting rules, *Social Choice and Welfare* **20**, 27–32.

Gelman, A., Katz, J. N. and Bafumi, J.: 2004, Standard voting power indices don't work: an empirical analysis, *British Journal of Political Science* **34**, 657–674.

Gelman, A., Katz, J. N. and Boscardin, W. J.: 2003, Estimating the probability of events that have never occurred: when is your vote decisive?, *American Journal of Political Science* **93**, 1–9.

Gelman, A., Katz, J. N. and Tuerlinckx, F.: 2002, The mathematics and statistics of voting power, *Statistical Science* **17**, 420–435.

Heard, A. and Swartz, T.: 1998, Empirical Banzhaf indices, *Public Choice* **97**, 701–707.

Kanazawa, S.: 1998, A brief note on a further refinement of the Condorcet jury theorem for heterogeneous groups, *Mathematical Social Sciences* **35**, 69–73.

Kaniovski, S.: 2008, The exact bias of the Banzhaf measure of power when votes are neither equiprobable nor independent, *Social Choice and Welfare* **forthcoming**.

Kaniovski, Y. M. and Pflug, G. C.: 2007, Risk assessment for credit portfolios: a coupled Markov chain model, *Journal of Banking & Finance* **31**, 2303–2323.

Ladha, K. K.: 1992, The Condorcet's jury theorem, free speech and correlated votes, *American Journal of Political Science* **36**, 617–634.

Langford, E., Schwertman, N. and Owens, M.: 2001, Is the property of being positively correlated transitive?, *The American Statistician* **55**, 322–325.

Laruelle, A. and Valenciano, F.: 2002, Assessment of voting situations: the probabilistic foundations. IVIE Working Papers, WP-AD 2002-22.

Leech, D.: 1990, Power indices and probabilistic voting assumptions, *Public Choice* **66**, 293–299.

Martin, R.: 1971, The concept of power: a critical defence, *British Journal of Sociology* **22**, 240–56.

Morriss, P.: 2002, *Power: a philosophical analysis*, Manchester University Press.

Napel, S. and Widgrén, M.: 2004, Power measurement as sensitivity analysis: a unified approach, *Journal of Theoretical Politics* **16**, 517 – 538.

Nitzan, S. and Paroush, J.: 1984, The significance of independent decisions in uncertain dichotomous choice situations, *Theory and Decision* **17**, 47–60.

Page, S. E.: 2006, Path dependence, *Quarterly Journal of Political Science* **1**, 87–115.

Penrose, L. S.: 1946, The elementary statistics of majority voting, *Journal of Royal Statistical Society* **109**, 53–57.

Shapley, L. S. and Shubik, M.: 1954, A method of evaluating the distribution of power in a committee system, *American Political Science Review* **48**, 787–792.

Straffin, P. D.: 1977, Homogeneity, independence and power indices, *Public Choice* **30**, 107–118.

Surowiecki, J.: 2005, *The wisdom of crowds. Why the many are smarter than the few*, Abacus.

Young, H. P.: 1988, Condorcet's theory of voting, *American Political Science Review* **82**, 1231–1244.

Table 1: Game: $\{2; 1, 1, 1\}$. Probabilities are rounded to three decimal places

	CASE 1						CASE 2						CASE 3					
Coalition	π	π_Y	π_N	L	N	N	π	π_Y	π_N	L	F	F	π	π_Y	π_N	L	C	C
1	0.250	0.500	0	1	1	1	0.313	0.625	0	1	1	1	0.188	0.375	0	1	1	1
2	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0
3	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
4	0.250	0.500	0	1	0	0	0.188	0.375	0	1	0	0	0.313	0.625	0	1	0	0
5	0.250	0	0.500	0	1	1	0.188	0	0.375	0	1	1	0.313	0	0.625	0	1	1
6	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
7	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
8	0.250	0	0.500	0	0	0	0.313	0	0.625	0	0	0	0.188	0	0.375	0	0	0
BzY				0	0.500	0.500				0	0.375	0.375				0	0.625	0.625
BzN				0	0.500	0.500				0	0.375	0.375				0	0.625	0.625

	CASE 4						CASE 5						CASE 6					
Coalition	π	π_Y	π_N	L	N	F	π	π_Y	π_N	L	N	C	π	π_Y	π_N	L	F	C
1	0.156	0.313	0	1	1	1	0.156	0.313	0	1	1	1	0.094	0.188	0	1	1	1
2	0.094	0.188	0	1	1	0	0.094	0.188	0	1	1	0	0.219	0.438	0	1	1	0
3	0.156	0.313	0	1	0	1	0.156	0.313	0	1	0	1	0.094	0.188	0	1	0	1
4	0.094	0.188	0	1	0	0	0.094	0.188	0	1	0	0	0.094	0.188	0	1	0	0
5	0.094	0	0.188	0	1	1	0.094	0	0.188	0	1	1	0.094	0	0.188	0	1	1
6	0.156	0	0.313	0	1	0	0.156	0	0.313	0	1	0	0.094	0	0.188	0	1	0
7	0.094	0	0.188	0	0	1	0.094	0	0.188	0	0	1	0.219	0	0.438	0	0	1
8	0.156	0	0.313	0	0	0	0.156	0	0.313	0	0	0	0.094	0	0.188	0	0	0
BzY				0.500	0.375	0.500				0.500	0.375	0.500				0.625	0.625	0.375
BzN				0.500	0.375	0.500				0.500	0.375	0.500				0.625	0.625	0.375

$p = 0.5$, $c_{N,N} = c_{F,F} = c_{C,C} = 1$, $c_{L,N} = c_{N,F} = c_{N,C} = 0$, $c_{L,F} = 0.25$, $c_{L,C} = c_{F,C} = -0.25$

Table 2: Game: $\{2; 1, 1, 1\}$. Probabilities are rounded to three decimal places

	CASE 7						CASE 8						CASE 9					
Coalition	π	π_Y	π_N	L	N	N	π	π_Y	π_N	L	F	F	π	π_Y	π_N	L	C	C
1	0.563	0.750	0	1	1	1	0.609	0.813	0	1	1	1	0.516	0.687	0	1	1	1
2	0	0	0	1	1	0	0	0	0	1	1	0	0	0	0	1	1	0
3	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
4	0.188	0.250	0	1	0	0	0.141	0.187	0	1	0	0	0.234	0.313	0	1	0	0
5	0.188	0	0.750	0	1	1	0.141	0	0.562	0	1	1	0.234	0	0.938	0	1	1
6	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
7	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
8	0.063	0	0.250	0	0	0	0.109	0	0.438	0	0	0	0.016	0	0.062	0	0	0
BzY				0	0.250	0.250				0	0.187	0.187				0	0.313	0.313
BzN				0	0.750	0.750				0	0.562	0.562				0	0.938	0.938

	CASE 10						CASE 11						CASE 12					
Coalition	π	π_Y	π_N	L	N	F	π	π_Y	π_N	L	N	C	π	π_Y	π_N	L	F	C
1	0.445	0.594	0	1	1	1	0.445	0.594	0	1	1	1	0.391	0.521	0	1	1	1
2	0.117	0.156	0	1	1	0	0.117	0.156	0	1	1	0	0.219	0.292	0	1	1	0
3	0.164	0.219	0	1	0	1	0.164	0.219	0	1	0	1	0.125	0.167	0	1	0	1
4	0.023	0.031	0	1	0	0	0.023	0.031	0	1	0	0	0.016	0.021	0	1	0	0
5	0.117	0	0.469	0	1	1	0.117	0	0.469	0	1	1	0.125	0	0.500	0	1	1
6	0.070	0	0.281	0	1	0	0.070	0	0.281	0	1	0	0.016	0	0.063	0	1	0
7	0.023	0	0.094	0	0	1	0.023	0	0.094	0	0	1	0.109	0	0.438	0	0	1
8	0.039	0	0.156	0	0	0	0.039	0	0.156	0	0	0	0	0	0	0	0	0
BzY				0.375	0.188	0.250				0.375	0.188	0.250				0.458	0.312	0.187
BzN				0.375	0.563	0.750				0.375	0.563	0.750				0.500	0.937	0.562

$p = 0.75$, $c_{N,N} = c_{F,F} = c_{C,C} = 1$, $c_{L,N} = c_{N,F} = c_{N,C} = 0$, $c_{L,F} = 0.25$, $c_{L,C} = c_{F,C} = -0.25$