A note on the probability of at least k successes in n correlated binary trials

Alexander Zaigraev* Serguei Kaniovski†

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Abstract

We obtain the probability distribution of the number of successes in a sequence of correlated binary trials as a function of marginal probabilities and correlation coefficients. It is based on Bahadur's representation of the joint probability distribution of correlated binary trials, truncated to second-order correlations. We provide new results, illustrating the use of this distribution in reliability theory and decision theory.

Key Words: correlated binary trials, k-out-of-n system, expert groups

1 Introduction

In this paper we generalize the binomial distribution to correlated binary variables based on Bahadur's (1961) representation of the joint probability distribution of n correlated binary variables, truncated to second-order correlations. The probability distribution of the number of successes in binary trials finds wide application in reliability theory and decision theory.

Reliability of k-out-of-n systems. Our first application studies the reliability of a k-out-of-n system with positively correlated component failures (Zuo and Tian 2010). Factors that may lead to dependent component performance include the influence of a common operating environment and failure cascades, if the failure of one component increases the strain on the remaining components. There exists ample empirical evidence for the positively correlated

*Nicolaus Copernicus University
Faculty of Mathematics and Computer Science
Chopin str. 12/18, 87-100 Toruń, Poland
Tel: +48 56 6112944
Email: alzaig@mat.uni.torun.pl
†Austrian Institute of Economic Research (WIFO)
P.O. Box 91, A-1103 Vienna, Austria
Tel: +43 1 7982601 231
Email: serguei.kaniovski@wifo.ac.at

component failures surveyed in Dhillon and Anude (1994), and a number of models study the effect of positive correlation between component failures on the reliability of a system (e.g., Littlewood 1996, Fiondella 2010).

We show that choosing an optimal degree of component redundancy k can mitigate the effect of correlation on the reliability of a k-out-of-n system with n equally reliable components. One additional advantage of a correlation-robust system is its predictability. The reliability of a correlation-robust system is as close as possible to that of an identical system with independently functioning components, and its reliability under independence can easily be established at the design stage of the system using the ordinary binomial distribution. The disadvantage is that minimizing the sensitivity to correlation may not maximize reliability, because the effect of correlation on reliability can be beneficial as well as detrimental.

Expertise of expert groups. The Condorcet Jury Theorem studies the collective expertise of a group of experts. The expertise of an expert group is the probability of it collectively making the correct decision under a given voting rule. The theorem rests on five assumptions: 1) the group must choose between two alternatives, one of which is correct; 2) the group makes its decision by voting under simple majority rule; 3) each expert is more likely than not to vote for the correct alternative; 4) all experts have equal probabilities (homogeneity); and 5) each expert makes his decision independently. The theorem states that any group comprising an odd number of experts greater than one is more likely to select the correct alternative than any single expert, and this likelihood becomes a certainty as the number of experts tends to infinity. A proof of this classic result in social sciences can be found in Young (1988) and Boland (1989).

We generalize the above model by allowing 1) heterogeneity of experts, 2) positive correlation between the votes, and 3) qualified majority rules. For analytical tractability, we assume that any two votes correlate with the same correlation coefficient. The conventional wisdom holds that the "groupthink" or "bandwagon" effect implied in positive correlation diminishes the collective competence of the group (Surowiecki 2005). We show that this effect can be positive or negative, and provide sufficient conditions for it to have a certain sign. This indicates that we can choose a voting rule k, for which the effect of positive correlation between the votes on the probability of a group collectively making correct decisions is positive.

2 The Bahadur representation

Let x_i be a realization of a binary random variable X_i, such that: $P(X_i = 1) = p_i$ and $P(X_i = 0) = 1 - p_i$, $p_i \in (0, 1)$ for all i. Let $\mathbf{x}$, the vector of n realizations, occur with the probability $\pi_{\mathbf{x}}$. Bahadur (1961) obtained the following representation of the joint probability distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$:

Theorem (Bahadur). *The joint probability distribution of n correlated binary random variables* *is uniquely determined by* n *marginal probabilities* (p_i) *and* $2^n - n - 1$ *correlation coefficients*

$$\begin{aligned}
&\textit{order 2} \quad c_{i,j} = E[Z_i Z_j] \quad \textit{for all} \quad 1 \le i < j \le n;\\
&\textit{order 3} \quad c_{i,j,r} = E[Z_i Z_j Z_r] \quad \textit{for all} \quad 1 \le i < j < r \le n;\\
&\qquad\qquad \ldots\\
&\textit{order } n \quad c_{1,2,\ldots,n} = E[Z_1 Z_2 \ldots Z_n],
\end{aligned}$$

where $Z_i = (X_i - p_i)/\sqrt{p_i(1-p_i)}$ *for* $i = 1, 2, \ldots, n$*, such that*

$$\pi_{\mathbf{x}} = \hat{\pi}_{\mathbf{x}} \Bigg(1 + \sum_{1 \le i < j \le n} c_{i,j} z_i z_j + \sum_{1 \le i < j < k \le n} c_{i,j,k} z_i z_j z_k + \cdots + c_{1,2,\ldots,n} z_1 z_2 \ldots z_n\Bigg). \tag{1}$$

Here

$$\hat{\pi}_{\mathbf{x}} = \prod_{i=1}^{n} p_i^{x_i} (1-p_i)^{1-x_i}$$

is the probability of $\boldsymbol{X} = \boldsymbol{x}$ *in the case of independence and* $z_i = (x_i - p_i)/\sqrt{p_i(1-p_i)}$ *is a realization of the random variable* Z_i *for* $i = 1, 2, \ldots, n$*.*

The second-order correlation coefficient is the Pearson product-moment correlation coefficient between two binary random variables. We will denote the $n \times n$ matrix of Pearson product-moment correlation coefficients by $\mathbf{C}$. Higher order correlation coefficients measure dependence between the general tuples of binary random variables. For a given n, there will be $\sum_{i=2}^{n} \binom{n}{i} = 2^n - n - 1$ correlation coefficients of all orders.

The Bahadur representation is sufficiently general to define the joint probability distribution of any vector of correlated binary variables. The drawback is the large number of parameters it therefore requires. Foreseeing this difficulty, Bahadur proposed truncating distribution (1) to second-order correlations. This is equivalent to assuming that all higher order correlations vanish. In this case distribution (1) becomes:

$$\pi_{\mathbf{x}} = \hat{\pi}_{\mathbf{x}} \Bigg(1 + \sum_{1 \le i < j \le n} c_{i,j} z_i z_j\Bigg). \tag{2}$$

Bahadur provided a lower bound on the smallest eigenvalue λ_{min} of the correlation matrix $\mathbf{C}$ required for (2) to be nonnegative for all $\mathbf{x}$:

$$\lambda_{min} \ge 1 - \frac{2}{\sum_{i=1}^{n} \beta_i}, \text{ where } \beta_i = \max\left\{\frac{p_i}{1-p_i}, \frac{1-p_i}{p_i}\right\} \text{ for each } i = 1, 2, \ldots, n.$$

Since in Bahadur's parametrization $\pi_{\mathbf{x}} \le 1$ by construction, we only need to ensure $\pi_{\mathbf{x}} \ge 0$. The above bound is sufficient but not necessary for $\pi_{\mathbf{x}}$ in (2) to be a distribution.

3 Probability of at least k successes

Let $t(\mathbf{x}) = \sum_{i=1}^{n} x_i$ denote the sum of coordinates of $\mathbf{x}$. We use distribution (2) to obtain the probability of at least k successes in n correlated binary trials:

$$P_n^k(\mathbf{p}, \mathbf{C}) = \sum_{\mathbf{x}:t(\mathbf{x})\geq k} \pi_{\mathbf{x}}, \quad \text{where} \quad k = 0, 1, \ldots, n, \tag{3}$$

under the assumption that all $\pi_{\mathbf{x}}$'s according to formula (2) are nonnegative. The cumulative distribution function of a generalized binomial distribution becomes:

$$P\{t(\mathbf{X}) \leq k\} = \begin{cases} 1 - P_n^{k+1}(\mathbf{p}, \mathbf{C}), & k = 0, 1, \ldots, n-1; \\ 1, & k = n. \end{cases}$$

Theorem. *The probability of at least k, $k = 0, 1, \ldots, n$, successes in n correlated binary trials is given by*

$$P_n^k(\mathbf{p}, \mathbf{C}) = P_n^k(\mathbf{p}, \mathbf{I}) + \prod_{h=1}^{n} p_h \sum_{1\leq i<j\leq n} c_{i,j}\alpha_i\alpha_j A_{i,j}^k(\boldsymbol{\alpha}), \tag{4}$$

where $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ is the vector of marginal probabilities, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_i = \sqrt{(1-p_i)/p_i}$ for $i = 1, 2, \ldots, n$, $\boldsymbol{C} = (c_{ij})$ the $n \times n$ correlation matrix, $\boldsymbol{I}$ the $n \times n$ identity matrix, and

$$A_{i,j}^k(\boldsymbol{\alpha}) = \begin{cases} 0, & k = 0; \\ \displaystyle\sum_{\substack{i_s\neq i,\, i_s\neq j \\ 1\leq i_1<\cdots<i_{n-k}\leq n}} \alpha_{i_1}^2 \ldots \alpha_{i_{n-k}}^2 - \sum_{\substack{j_s\neq i,\, j_s\neq j \\ 1\leq j_1<\cdots<j_{n-k-1}\leq n}} \alpha_{j_1}^2 \ldots \alpha_{j_{n-k-1}}^2, & k = 1, 2, 3, \ldots, n-1; \\ 1, & k = n. \end{cases}$$

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector of binary outcomes. We begin the proof by rewriting Bahadur's expression (2) for the joint distribution in terms of α_i. Since in (2) $z_i = (-1)^{1-x_i}\alpha_i^{2x_i-1}$ for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\pi_{\mathbf{x}} &= \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} + \prod_{h=1}^{n} p_h \prod_{s=1}^{n} \alpha_s^{2(1-x_s)} \sum_{1\leq i<j\leq n} c_{i,j} z_i z_j = \\
&= \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} + \prod_{h=1}^{n} p_h \prod_{s=1}^{n} \alpha_s^{2(1-x_s)} \sum_{1\leq i<j\leq n} c_{i,j}(-1)^{2-x_i-x_j}\alpha_i^{2x_i-1}\alpha_j^{2x_j-1} = \\
&= \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} + \prod_{h=1}^{n} p_h \sum_{1\leq i<j\leq n} c_{i,j}\alpha_i\alpha_j(-1)^{x_i+x_j} \prod_{s\neq i,\, s\neq j} \alpha_s^{2(1-x_s)}.
\end{aligned}$$

In view of the above formula for $\pi_{\mathbf{x}}$, the probability of exactly k successes in n trials can be written as

$$\sum_{\mathbf{x}:\,t(\mathbf{x})=k}\prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i}+\prod_{h=1}^{n}p_h\sum_{1\le i<j\le n}c_{i,j}\alpha_i\alpha_j\sum_{\mathbf{x}:\,t(\mathbf{x})=k}(-1)^{x_i+x_j}\prod_{s\neq i,\,s\neq j}\alpha_s^{2(1-x_s)}. \tag{5}$$

Let

$$B_{i,j}^k(\boldsymbol{\alpha})=\sum_{\mathbf{x}:\,t(\mathbf{x})=k}(-1)^{x_i+x_j}\prod_{s\neq i,\,s\neq j}\alpha_s^{2(1-x_s)} \text{ for } i,j=1,\dots,n,\ 1\le i<j\le n. \tag{6}$$

Consider $B_{i,j}^k(\boldsymbol{\alpha})$ for all k such that $2\le k\le n-2$. For the remaining values of k, the term $B_{i,j}^k(\boldsymbol{\alpha})$ will be studied below. Split the sum (6) in three summands: first, over all the vectors $\mathbf{x}$, such that $x_i=1$ and $x_j=1$; second, over all the $\mathbf{x}$, such that $x_i=1$ and $x_j=0$, or $x_i=0$ and $x_j=1$; and, third, over all the $\mathbf{x}$, such that $x_i=0$ and $x_j=0$. To simplify notations, we drop the argument $\boldsymbol{\alpha}$. We have,

$$B_{i,j}^k=C_{n-k}-2C_{n-k-1}+C_{n-k-2},\ \text{where } C_m=\sum_{\substack{i_s\neq i,\,i_s\neq j\\ 1\le i_1<\dots<i_m\le n}}\alpha_{i_1}^2\dots\alpha_{i_m}^2.$$

For the remaining cases $k=0$, $k=1$, $k=n-1$, $k=n$, putting $C_0=1$,

$$B_{i,j}^0=C_{n-2},\quad B_{i,j}^1=-2C_{n-2}+C_{n-3},\quad B_{i,j}^{n-1}=C_1-2C_0,\quad B_{i,j}^n=C_0.$$

Note that

$$B_{i,j}^n=C_0,\quad B_{i,j}^n+B_{i,j}^{n-1}=C_1-C_0,\quad B_{i,j}^n+B_{i,j}^{n-1}+B_{i,j}^{n-2}=C_2-C_1; \tag{7}$$

$$\dots$$

$$B_{i,j}^n+\dots+B_{i,j}^k=C_{n-k}-C_{n-k-1}, \tag{8}$$

$$\dots$$

$$B_{i,j}^n+\dots+B_{i,j}^2=C_{n-2}-C_{n-3},\quad B_{i,j}^n+\dots+B_{i,j}^1=-C_{n-2},\quad B_{i,j}^n+\dots+B_{i,j}^0=0. \tag{9}$$

Combining (5) and (6) yields

$$P_n^k(\mathbf{p},\mathbf{C})=P_n^k(\mathbf{p},\mathbf{I})+\prod_{h=1}^{n}p_h\sum_{1\le i<j\le n}c_{i,j}\alpha_i\alpha_j\sum_{l=k}^{n}B_{i,j}^l.$$

Substituting $\sum_{l=k}^n B_{i,j}^l$ from (7)-(9) in the above formula furnishes the theorem. □

When computing the cumulative distribution function using (4), note that if we write $A_{i,j}^k=\mathcal{B}_{i,j}^k-\mathcal{C}_{i,j}^k$, then $\mathcal{B}_{i,j}^1=\mathcal{C}_{i,j}^{n-1}=0$ and $\mathcal{B}_{i,j}^k=\mathcal{C}_{i,j}^{k-1}$. The last equality effectively halves the number of sums required to compute $A_{i,j}^k$. Since for given i and j, computing $A_{i,j}^k$ for all k requires computing $(n-2)$ sums $\mathcal{B}_{i,j}^k$ and $\mathcal{C}_{i,j}^k$, the number of such sums for all k and all $i < j$ equals $n(n-1)(n-2)/2$. The above computations exclude the trivial cases $k = 0$ and $k = n$. For all k, the total number of summands in the second term of (4) equals $n(n-1)^2/2$.

4 Robustness of a k-out-of-n system to correlation

A k-out-of-n system consists of n components, such that each component is either functional, or has failed. A k-out-of-n system functions if k or more of its components function. Let component i's state be a realization x_i of a binary random variable X_i, such that

$$x_i = \begin{cases} 1, & \text{if component } i \text{ functions;} \\ 0, & \text{if component } i \text{ fails.} \end{cases}$$

The reliability of component i is measured by its probability of being functional $p_i = P(X_i = 1)$.

A vector of states $\mathbf{x} = (x_1, \ldots, x_n)$ is called a system profile. There will be 2^n such profiles. The structure function of a k-out-of-n system is defined as

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \geq k; \\ 0, & \text{if } \sum_{i=1}^n x_i < k. \end{cases}$$

Let $\pi_{\mathbf{x}}$ be the probability of $\mathbf{X} = \mathbf{x}$, and $t(\mathbf{x})$ be the number of functioning components. The reliability of a system is defined as the probability that the system will function:

$$R_n^k(\mathbf{p}, \mathbf{C}) = E[\phi(\mathbf{X})] = P(\phi(\mathbf{X}) = 1) = P_n^k(\mathbf{p}, \mathbf{C}), \text{ where } k = 1, 2, \ldots, n.$$

The k-out-of-n notation conveys the arrangement of system components. In a parallel system $(k = 1)$ at least one of the n components must function in order for the system to function. A functioning series system $(k = n)$ requires all of its components to be functional.

Fiondella (2010) suggests using a Birnbaum measure to assess the sensitivity of a system's reliability to correlation. The Birnbaum measure for correlation coefficients is defined as:

$$BM_n^k(\mathbf{p}, \mathbf{C}) = \frac{\partial\, R_n^k(\mathbf{p}, \mathbf{C})}{\partial\, c_{i,j}}.$$

This measure is useful if the system designer can control correlation so as to maximize reliability. Harnessing correlation may not be feasible if the dependency in component performance owes to the influence of an adverse operating environment (Lindley and Singpurwalla 2002). In this case, it would be desirable to mitigate the effect of correlation on system reliability by minimizing the absolute value of the Birnbaum measure $|BM_n^k(\mathbf{p}, \mathbf{C})|$.

In the following analysis, the design of the system is given by k or the degree of redundancy in a system comprising n equally reliable components, so that $p_i = p \in (0, 1)$ for all $i = 1, 2, \ldots, n$.

We propose choosing k that minimizes or sometimes even negates the effect of correlation on system reliability. Formula (4) furnishes that $R_n^k(\mathbf{p}, \mathbf{C})$ does not depend on $\mathbf{C}$ if $A_{i,j}^k(\boldsymbol{\alpha}) = 0$ for all $i < j$. By assumption, $p_i = p$, so that $A_{i,j}^k(\boldsymbol{\alpha}) = A^k(\alpha)$. Consequently, $R_n^k(p, \mathbf{C})$ does not depend on $\mathbf{C}$ if $A^k(\alpha) = 0$. This is not possible in a series system, where $k = n$, or in a parallel system, where $k = 1$. For any intermediate system, the effect of correlation on reliability is minimal for k that minimizes $|A^k(\alpha)|$. Solving the equation $\binom{n-2}{n-\kappa}\alpha^{2(n-\kappa)} = \binom{n-2}{n-\kappa-1}\alpha^{2(n-\kappa-1)}$ with respect to k yields $\kappa = p(n-1)+1$. If κ is an integer, then choosing $k = \kappa$ would make the k-out-of-n system immune to correlation. If κ is not an integer, two feasible correlation robust k-out-of-n systems can be defined using the greatest integer smaller to or equal to κ ($\lfloor\kappa\rfloor$) and the smallest integer larger to or equal to κ ($\lceil\kappa\rceil$). Using the identities $\lfloor\kappa\rfloor = \lceil\kappa\rceil - 1$ and $x\binom{n-1}{x} = (n-x)\binom{n-1}{x-1}$ one can show that

$$|A^{\lfloor\kappa\rfloor}(\alpha)| < |A^{\lceil\kappa\rceil}(\alpha)| \iff (1-p)\frac{\kappa - \lfloor\kappa\rfloor}{n - \lfloor\kappa\rfloor} < p\frac{\lfloor\kappa\rfloor - \kappa + 1}{\lfloor\kappa\rfloor}.$$

The above inequality holds for $p \geq \frac{n-2}{n-1}$. It follows that when component reliability is sufficiently high, the correlation robust k equals $\lfloor p(n-1)+1\rfloor$.

5 The effect of correlation on competence of an expert group

Take an odd number of experts n ($n \geq 3$). Each expert is competent in the sense of being more likely than not to decide correctly, but some experts are better than others. Both ideas are reflected in the assumption $p_i > 0.5$ for all i. The opinions of any two experts are correlated with the same positive correlation coefficient $c_{i,j} = c > 0$ for all $i < j$.

Individual opinions are aggregated into a collective decision using a voting rule k, such that $k = (n+1)/2, \dots, n$. Simple majority rule requires at least $k = (n+1)/2$ votes in favor of a decision being passed, whereas setting $k = n$ stipulates unanimity. The positive effect of correlation on the collective expertise under unanimity rule is evident because, as we noted in our theorem, $A_{i,j}^n(\boldsymbol{\alpha}) = 1$ for all $i < j$ in (4). The following lemma establishes the negative effect of correlation under simple majority rule.

Lemma. *If $p_i \in (0.5, 1)$ for $i = 1, \dots, n$, then $A_{i,j}^{(n+1)/2}(\boldsymbol{\alpha}) < 0$ for any $i, j = 1, \dots, n$, $i < j$.*

Proof.

$$A_{i,j}^{(n+1)/2}(\boldsymbol{\alpha}) = \sum_{\substack{i_s \neq i,\, i_s \neq j \\ 1 \leq i_1 < \cdots < i_{(n-1)/2} \leq n}} \alpha_{i_1}^2 \dots \alpha_{i_{(n-1)/2}}^2 - \sum_{\substack{j_s \neq i,\, j_s \neq j \\ 1 \leq j_1 < \cdots < j_{(n-3)/2} \leq n}} \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2.$$

Both sums comprise an equal number of summands, $\binom{n-2}{(n-1)/2}$ and $\binom{n-2}{(n-3)/2}$ respectively.

It suffices to prove the lemma for $i = 1$ and $j = 2$. The validity of the lemma for an arbitrary pair of indexes $i < j$ follows because their choice is not essential for the proof.

Each term of the second sum corresponds to a set of indexes $A_{j_1\dots j_{(n-3)/2}} = \{j_1, \dots, j_{(n-3)/2}\}$. Let $\widetilde{A}_{j_1\dots j_{(n-3)/2}} = \{3, \dots, n\} \setminus A_{j_1\dots j_{(n-3)/2}}$. Since the set $\widetilde{A}_{j_1\dots j_{(n-3)/2}}$ contains $(n-1)/2$ indexes,

$$\sum_{3\le j_1<\dots<j_{(n-3)/2}\le n} \sum_{m\in \widetilde{A}_{j_1\dots j_{(n-3)/2}}} \alpha_m^2 \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2 = \frac{n-1}{2} \sum_{3\le i_1<\dots<i_{(n-1)/2}\le n} \alpha_{i_1}^2 \dots \alpha_{i_{(n-1)/2}}^2.$$

It is evident that the difference

$$\sum_{3\le i_1<\dots<i_{(n-1)/2}\le n} \alpha_{i_1}^2 \dots \alpha_{i_{(n-1)/2}}^2 - \sum_{3\le j_1<\dots<j_{(n-3)/2}\le n} \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2 =$$

$$= \frac{2}{n-1} \sum_{3\le j_1<\dots<j_{(n-3)/2}\le n} \sum_{m\in \widetilde{A}_{j_1\dots j_{(n-3)/2}}} \alpha_m^2 \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2 - \sum_{3\le j_1<\dots<j_{(n-3)/2}\le n} \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2 =$$

$$= \frac{2}{n-1} \sum_{3\le j_1<\dots<j_{(n-3)/2}\le n} \alpha_{j_1}^2 \dots \alpha_{j_{(n-3)/2}}^2 \left[\sum_{m\in \widetilde{A}_{j_1\dots j_{(n-3)/2}}} \alpha_m^2 - \frac{n-1}{2} \right] < 0.$$

The last step follows since $p_i \in (0.5, 1)$ for $i = 1, \dots, n$ implies $\alpha_i \in (0, 1)$. □

The effect of a common positive correlation on the collective expertise is negative under simple majority rule and positive under unanimity rule. This points towards the existence of intermediate rules $\frac{n+1}{2} < k < n$, for which the effect of correlation changes its sign. Although it is not possible to obtain a general result on the sign of the effect for intermediate voting rules, the following condition is sufficient for this effect to have a certain sign.

Sufficient Condition 1. *If the sum of any $k-1$ of reciprocal probabilities $\{p_s^{-1}\}$ is not smaller than $n-1$, then $\forall i, j$, $i < j$, we have $A_{i,j}^m(\boldsymbol{\alpha}) \ge 0$, $\forall m \ge k$. If the sum of any $k-1$ of reciprocal probabilities $\{p_s^{-1}\}$ is not greater than $n-1$, then $\forall i, j$, $i < j$, we have $A_{i,j}^m(\boldsymbol{\alpha}) \le 0$, $\forall m \le k$.*

The above sufficient condition has clear implications for an optimal choice of the voting rule in relationship to the harmonic mean and the range of individual competences $\{p_i\}$.

Sufficient Condition 1 implies inequalities on the harmonic mean $H = n(\sum_{i=1}^n p_i^{-1})^{-1}$. If the sum of any $k-1$ of $\{p_s^{-1}\}$ is not smaller than $n-1$, then $H \le \frac{k-1}{n-1}$; if the sum of any $k-1$ of $\{p_s^{-1}\}$ is not greater than $n-1$, then $H \ge \frac{k-1}{n-1}$.

Sufficient Condition 2. *If $p_i \in (0.5, \frac{k-1}{n-1}]$, $\forall i$, then $A_{i,j}^m(\boldsymbol{\alpha}) \ge 0$, $\forall m \ge k$, $\forall i, j$, $i < j$. If $p_i \in [\frac{k-1}{n-1}, 1)$, $\forall i$, then $A_{i,j}^m(\boldsymbol{\alpha}) \le 0$, $\forall m \le k$, $\forall i, j$, $i < j$.*

Sufficient Condition 2 follows from Sufficient Condition 1, as does the following condition:

Sufficient Condition 3. *If there exist $\frac{n+1}{2} \le m < l \le n$ such that all $\{p_i\}$ lie within the interval $[\frac{m-1}{n-1}, \frac{l-1}{n-1}]$, then $A_{i,j}^{(n+1)/2}(\boldsymbol{\alpha}) < 0$, $\dots$, $A_{i,j}^m(\boldsymbol{\alpha}) \le 0, A_{i,j}^l(\boldsymbol{\alpha}) \ge 0, \dots, A_{i,j}^n(\boldsymbol{\alpha}) > 0$, $\forall i, j$, $i < j$.*

Proof. Since Sufficient Condition 3 follows from Sufficient Condition 2, and the latter follows from Sufficient Condition 1, we only need to prove Sufficient Condition 1.

Let i, j, such that $i < j$, be given. If $k = n-1$, then $A^{n-1}_{i,j}(\boldsymbol{\alpha}) = \sum_{s\neq i,\, s\neq j} \alpha_s^2 - 1$,

$$A^{n-1}_{i,j}(\boldsymbol{\alpha}) \geq 0 \Longleftrightarrow \sum_{s\neq i,\, s\neq j} p_s^{-1} \geq n-1 \quad \text{and} \quad A^{n-1}_{i,j}(\boldsymbol{\alpha}) \leq 0 \Longleftrightarrow \sum_{s\neq i,\, s\neq j} p_s^{-1} \leq n-1.$$

To prove Sufficient Condition 1 for the remaining values of k, we proceed as in the proof of the lemma. Without any loss of generality, assume that $i = 1$ and $j = 2$. If $k = n-2$, then

$$A^{n-2}_{1,2}(\boldsymbol{\alpha}) = \sum_{3\leq i_1<i_2\leq n} \alpha_{i_1}^2 \alpha_{i_2}^2 - \sum_{s=3}^{n} \alpha_s^2.$$

For any fixed s, consider the set of indexes $\widetilde{A}_s = \{3, \ldots, n\} \setminus \{s\}$. We have,

$$\sum_{3\leq i_1<i_2\leq n} \alpha_{i_1}^2 \alpha_{i_2}^2 - \sum_{s=3}^{n} \alpha_s^2 = \frac{1}{2}\sum_{s=3}^{n} \sum_{m\in\widetilde{A}_s} \alpha_m^2 \alpha_s^2 - \sum_{s=3}^{n} \alpha_s^2 = \frac{1}{2}\sum_{s=3}^{n} \alpha_s^2 \Bigg[\sum_{m\in\widetilde{A}_s} \alpha_m^2 - 2 \Bigg].$$

It remains to note that since the set $\widetilde{A}_s$ contains $n-3 = k-1$ indexes,

$$\sum_{m\in\widetilde{A}_s} \alpha_m^2 - 2 = \sum_{m\in\widetilde{A}_s} p_m^{-1} - (n-1).$$

Thus,

$$A^{n-2}_{1,2}(\boldsymbol{\alpha}) \geq 0 \Longleftrightarrow \sum_{m\in\widetilde{A}_s} p_m^{-1} \geq n-1 \quad \text{and} \quad A^{n-2}_{1,2}(\boldsymbol{\alpha}) \leq 0 \Longleftrightarrow \sum_{m\in\widetilde{A}_s} p_m^{-1} \leq n-1.$$

The same reasoning can be repeated for any k such that $\frac{n+1}{2} < k < n$. □

References

Bahadur, R. R.: 1961, A representation of the joint distribution of responses to n dichotomous items, *in* H. Solomon (ed.), *Studies in item analysis and prediction*, Stanford University Press, pp. 158–168.

Boland, P. J.: 1989, Majority systems and the Condorcet jury theorem, *The Statistician* **38**, 181–189.

Dhillon, B. and Anude, O.: 1994, Common-cause failures in engineering systems: A review, *International Journal of Reliability, Quality and Safety Engineering* **1**, 103–129.

Fiondella, L.: 2010, Reliability and sensitivity analysis of coherent systems with negatively correlated component failures, *International Journal of Reliability, Quality and Safety Engineering* **17**, 505–529.

Lindley, D. V. and Singpurwalla, N. D.: 2002, On exchangeable, causal and cascading failures, *Statistical Science* **17**, 209–219.

Littlewood, B.: 1996, The impact of diversity upon common mode failures, *Reliability Engineering and System Safety* **51**, 101–113.

Surowiecki, J.: 2005, *The wisdom of crowds. Why the many are smarter than the few*, Abacus.

Young, H. P.: 1988, Condorcet's theory of voting, *American Political Science Review* **82**, 1231–1244.

Zuo, M. J. and Tian, Z.: 2010, k-out-of-n systems, *in* J. J. Cochran, L. A. Cox, P. Keskinocak, J. P. Kharoufeh and J. C. Smith (eds), *Wiley Encyclopedia of Operations Research and Management Science*, John Wiley & Sons, Inc.