The average representation – a cornucopia of power indices?

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Abstract

We show how to construct power indices that respect proportionality between power and weight from average representations of a game. If restricted to account for the lack of power of dummy voters, average representations become coherent measures of voting power, such that the resulting power distributions are proportional to the distribution of weights. Imposing other restrictions may lead to a whole family of such power indices.

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1 Introduction

The observation that the distribution of power in weighted voting games differs from the distribution of voting weights has motivated the development of a theory of power measurement. A famous example considers three voters, having 49, 49, and 2 votes. The motion is passed if the total number of votes in favor exceeds 50. Since any two voters can pass the motion, any reasonable power index assigns equal power to all three voters.

The power distribution in the above example markedly differs from the relative weight distribution: $\left\|\left(\frac{49}{100},\frac{49}{100},\frac{2}{100}\right)-\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\|_1=\frac{47}{75}\approx 0.63$. The main reason

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for this disagreement lies in the fact that voting weights are not unique. For three voters having 100 votes in total, there will be 1176 integer-valued weight distributions consistent with the above power vector. An example is given by the weights 34, 33, 33 and a quota of 60. There would be 13872 possibilities to represent the game if the integer quota were part of the specification.

Having plenty of representations to choose from, can we choose voting weights that accurately reflect power measured by some index? The theoretical literature shows that, in general, we cannot, although there are particular cases in which it may be possible. Recently (Houy & Zwicker, 2014) have characterized a class of weighted majority games that admit a representation using their respective Banzhaf distribution. For the nucleolus, we known that $(q(x^\star, v), x^\star)$ is a representation of a constant-sum weighted majority game, where $x^\star(v)$ denotes the nucleolus of v, and $q(x^\star, v)$ denotes the corresponding maximum excess (Maschler *et al.*, 2013, Theorem 20.52). If the weights are close to the average weight of the voters, then the nucleolus is close to the relative weight distribution; the two may even coincide under certain conditions, see (Kurz *et al.*, 2014). The existing power indices are not representation-compatible. One exception is the recently introduced minimum sum representation (MSR) index (Freixas & Kaniovski, 2014), which has been specifically designed to be proportional to voting weights in the minimum sum representation of a weighted game. Also the Colomer index (Colomer & Martinez, 1995) uses weights of a majority game in its specification, but the index depends on the given representation, instead of the underlying simple game.

We show how to define representation compatible power indices for weighted majority games, and to make them fulfill certain useful properties. The indices are based on average representations, which are already close to being coherent power indices, but which fail the essential dummy property. If restricted to account for the lack of power of dummy voters, average representations become coherent measures of voting power, such that the resulting power distributions are proportional to the distribution of weights in the average representation. In the final section of the paper, we discuss how imposing other restrictions may lead to other indices, and we discuss a convergence result for integer-valued representations.

2 Games and representations

A simple game v is a mapping $v : 2^N \to \{0, 1\}$, where $N = \{1, \ldots, n\}$ is the set of voters, such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.

The subsets $S \subseteq N$ are called coalitions of v. We call a coalition S winning if $v(S) = 1$, and losing otherwise. If S is a winning coalition and none of its proper subsets is winning, it is called a minimal winning coalition. Similarly, if T is a losing coalition and none of its proper supersets is losing, it is called a maximal losing coalition. The set of minimal winning coalitions $\mathcal{W}^m$, or the set of maximal losing coalitions $\mathcal{L}^m$, uniquely define a simple game. A voter $i \in N$ with $v(S) = v(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$ is called a dummy.

A weighted majority game is a simple game v, such that there exist real numbers $w_1, \ldots, w_n \geq 0$ and $q > 0$ with $\sum_{s \in S} w_s \geq q$ for all winning coalitions $S \subseteq N$ and $\sum_{s \in T} w_s < q$ for all losing coalitions $T \subseteq N$. We write $v = [q; w_1, \ldots, w_n]$, where we call $(q, w_1, \ldots, w_n)$ a representation of v. A weight vector $(w_1, \ldots, w_n)$ is called feasible for v, if there exists a quota q such that $(q; w_1, \ldots, w_n)$ is a representation of v. For our initial example $[2; 1, 1, 1]$, the weight vector $(49, 49, 2)$ is feasible, while $(50, 25, 25)$ is not feasible.

Lemma 1 *Each weighted majority game v admits a representation $(q, w_1, \ldots, w_n)$ with $w_1, \ldots, w_n \geq 0$, $q > 0$, and*

(1) $\sum_{i=1}^n w_i = 1$, $q \in (0, 1]$;

(2) $\sum_{i=1}^n w_i = 1$, $q \in (0, 1]$ and $w_i = 0$ for all dummies $i \in N$;

(3) $q \in \mathbb{N}$, $w_i \in \mathbb{N}$.

We call a representation satisfying the conditions of (1) a normalized representation, and those satisfying the conditions of (3) an integer representation. A representation with $w_i = 0$ for all dummies $i \in N$ is called dummy-revealing.[1]

Algorithmic checks and descriptions of whether a given simple game is weighted have been studied extensively in the literature, see e.g. (Taylor & Zwicker, 1999).

Lemma 2 *The set of all normalized weight vectors $w \in \mathbb{R}^n_{\geq 0}$, $\sum_{i=1}^n w_i = 1$ being feasible for a given weighted majority game v is given by the intersection $\sum_{i \in S} w_i > \sum_{i \in T} w_i$ for all pairs (S, T), where $S \subseteq \mathcal{W}^m$ and $T \subseteq \mathcal{L}^m$.*

Lemma 3 *The set of all normalized representations $(q, w) \in \mathbb{R}^{n+1}_{\geq 0}$, $q \in (0, 1]$, $\sum_{i=1}^n w_i = 1$ representing a given weighted majority game v is given by the intersection $\sum_{i \in S} w_i \geq q$, $\sum_{i \in T} w_i < q$ for all $S \subseteq \mathcal{W}^m$ and $T \subseteq \mathcal{L}^m$.*

[1]The problem of checking whether a voter is a dummy in a general (integer) representation it coNP-complete (Chalkiadakis *et al.*, 2011, Theorem 4.4).

3 Power indices

Let $\mathcal{S}_n$ denote the set of simple games on n voters, and $\mathcal{W}_n$ the set of weighted majority games on n voters. A power index for $\mathcal{C} \in \{\mathcal{S}_n, \mathcal{W}_n \mid n \in \mathbb{N}\}$ is a mapping $g : \mathcal{C} \to \mathbb{R}^n$, where n denotes the number of voters in each game of $\mathcal{C}$. Usually, we define a vector-valued power index by defining its elements g_i, the power of a voter $i \in N$. The Shapley-Shubik index is given by

$$SSI_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - 1 - |S|)!}{|N|!} \cdot (v(S \cup \{i\}) - v(S)).$$

The Shapley-Shubik index is symmetric, positive, efficient, and satisfies the dummy property.

Definition 4 *Let $g : \mathcal{C} \to \mathbb{R}^{|N|} = (g_i)_{i \in N}$ be a power index for $\mathcal{C}$. We say that*

(1) g is symmetric: if for all $v \in \mathcal{C}$ and any bijection $\tau : N \to \tau$ we have $g_{\tau(i)}(\tau v) = g_i(v)$, where $\tau v(S) = v(\tau(S))$ for all $S \subseteq N$;

(2) g is positive: if for all $v \in \mathcal{C}$ we have $g_i(v) \geq 0$ and $g(v) \neq 0$;

(3) g is efficient: if for all $v \in \mathcal{C}$ we have $\sum_{i=1}^{n} g_i(v) = 1$;

(4) g satisfies the dummy property: if for all $v \in \mathcal{C}$ and all dummies i of v we have $g_i(v) = 0$.

We consider power indices that can be defined on the set of weighted games. Having proportionality of weights and power in mind, we define:

Definition 5 *A power index $g : \mathcal{W}_n \to \mathbb{R}^n$ for weighted majority games on n voters is called representation compatible if $(g_1(v), \ldots, g_n(v))$ is feasible for all $v \in \mathcal{W}_n$.*

We remark that the Shapley-Shubik index is representation compatible for $\mathcal{W}_n$ if and only if $n \leq 3$. Below, we list the weighted majority games with up to 3 voters (in minimum sum integer representation), and the representation given by the Shapley-Shubik vector.

$$[1;1] = [1;1] \quad [1;1,0,0] = \left[\tfrac{6}{6};\tfrac{6}{6},\tfrac{0}{6},\tfrac{0}{6}\right] \quad [2;1,1,1] = \left[\frac{4}{6};\frac{2}{6},\frac{2}{6},\frac{2}{6}\right]$$

$$[1;1,0] = \left[\frac{2}{2};\frac{2}{2};\frac{0}{2}\right] \quad [1;1,1,0] = \left[\tfrac{3}{6};\tfrac{3}{6},\tfrac{3}{6},\tfrac{0}{6}\right] \quad [3;1,1,1] = \left[\frac{6}{6};\frac{2}{6},\frac{2}{6},\frac{2}{6}\right]$$

$$[1;1,1] = \left[\frac{1}{2};\frac{1}{2},\frac{1}{2}\right] \quad [2,1,1,0] = \left[\tfrac{6}{6};\tfrac{3}{6},\tfrac{3}{6},\tfrac{0}{6}\right] \quad [3;2,1,1] = \left[\frac{5}{6};\frac{4}{6},\frac{1}{6},\frac{1}{6}\right]$$

$$[2,1,1] = \left[\frac{2}{2};\frac{1}{2},\frac{1}{2}\right] \quad [1;1,1,1] = \left[\tfrac{2}{6};\tfrac{2}{6},\tfrac{2}{6},\tfrac{2}{6}\right] \quad [2;2,1,1] = \left[\frac{2}{6};\frac{4}{6},\frac{1}{6},\frac{1}{6}\right]$$

For $n \geq 4$, consider the example $v = [3;2,1,1,1,0,\ldots,0]$ with $n-4$ dummies. The Shapley-Shubik index of v is given by $\left(\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},0,\ldots,0\right)$. Since $\{2,3,4\}$ is a winning coalition with weight $\frac{1}{2}$, and $\{1\}$ is a losing coalition with an equal weight, the Shapley-Shubik vector cannot be representation-compatible.

4 Representation-compatible power indices

Given a set of representations of the same weighted majority game v, each convex combination also gives a representation of v. A simple way of constructing a representation-compatible power index is to specify the set of representations and the weights of the convex combination.

4.1 The average weight index

Definition 6 *The average weight index of a weighted majority game v is the average of all normalized*[2] *weight vectors which are feasible for v.*

For $[3;2,1,1]$ we have already mentioned the sets of minimal winning and maximal losing coalitions. Applying Lemma 2 gives the constraints

$$\begin{aligned} w_1 + w_2 > w_1 &\iff w_2 > 0 \\ w_1 + w_3 > w_1 &\iff w_3 > 0 \\ w_1 + w_2 > w_2 + w_3 &\iff w_1 > w_3 \\ w_1 + w_3 > w_2 + w_3 &\iff w_1 > w_2, \end{aligned}$$

[2]Taking all weight vectors instead of the normalized ones does not make a difference.

in addition to $w_1, w_2, w_3 \geq 0$ and $w_1 + w_2 + w_3 = 1$. Eliminating the variable w_3 via $w_3 = 1 - w_1 - w_2$ and removing the redundant constraints leaves

$$\begin{aligned} w_2 > 0 &\iff w_2 > 0 \\ 1 - w_1 - w_2 > 0 &\iff w_2 < 1 - w_1 \\ w_1 > 1 - w_1 - w_2 &\iff w_2 > 1 - 2w_1 \\ w_1 > w_2 &\iff w_2 < w_1 \end{aligned}$$

Since we need $1 - 2w_1 < w_1$ and $1 - w_1 > 0$, we have $w_1 \in \left(\frac{1}{3}, 1\right)$. For $w_1 \in \left(\frac{1}{3}, \frac{1}{2}\right)$ the conditions condense to $w_2 \in \left(1 - 2w_1, w_1\right)$ and for $w_1 \in \left[\frac{1}{2}, 1\right)$ the conditions condense to $w_2 \in \left(0, 1 - w_1\right)$.

The (scaled) average weight for voter 1 is given by

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} w_1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} w_1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 = \frac{1}{54} + \frac{1}{12} = \frac{11}{108}.$$

For voter 2 we similarly obtain

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} w_2 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} w_2 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 = \frac{1}{48} + \frac{5}{432} = \frac{7}{216}.$$

Since the volume of the feasible region is given by

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} 1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} 1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 = \frac{1}{8} + \frac{1}{24} = \frac{1}{6},$$

we have

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} w_3 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} w_3 \,\mathrm{d}\, w_2 \,\mathrm{d}\, w_1 = \frac{1}{6} - \frac{11}{108} - \frac{7}{216} = \frac{7}{216}.$$

Normalizing, or dividing by the volume of the feasible region, yields the power distribution $\left(\frac{11}{18}, \frac{7}{36}, \frac{7}{36}\right)$, with a norm-1-distance of $\frac{1}{9}$ to the respective Shapley-Shubik vector $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

The above set of inequalities defines a polyhedron in Euclidean space:

$$P = \left\{(w_1, w_2) \in \mathbb{R}^2 \mid w_2 \geq 0, w_2 \leq 1 - w_1, w_2 \geq 1 - 2w_1, w_2 \leq w_1\right\}.$$

Note that we have replaced strict inequalities by the corresponding non-strict inequalities, and considered the integral $\int_p w_1 \,\mathrm{d}\, w_1$, $\int_p w_2 \,\mathrm{d}\, w$, and $\int_p 1 \,\mathrm{d}\, w$. This modification is permitted since in general the polytope P (after the elimination of variable w_n) is full-dimensional, i.e. it has dimension $n-1$.

Lemma 7 *For each weighted majority game v there exist positive real numbers $\tilde{q}, \tilde{w}_1, \dots, \tilde{w}_{n-1}$ and a parameter $\alpha > 0$ such that*

$$\left(\tilde{q} + \delta_0, \tilde{w}_1 + \delta_1, \dots, \tilde{w}_{n-1} + \delta_{n-1}, 1 - \sum_{i=1}^{n-1} (\tilde{w}_i + \delta_i)\right) \tag{1}$$

is a normalized representation of v for all $\delta_i \in [-\alpha, \alpha]$, $0 \le i \le n-1$.

Proof. Let $(q, w_1, \dots, w_n)$ be an integer representation of v. The weight of each winning coalition is at least q, and the weight of each losing coalition is at most $q-1$. Since $\big((n+1)q, (n+1)w_1+1, \dots, (n+1)w_n\big)$ is also an integer representation of v, we additionally assume w.l.o.g. that $w_i \ge 1$ for all $1 \le i \le n$. One can easily check that also $\left(q - \frac{2}{5} + \tilde{\delta}_0, w_1 + \tilde{\delta}_1, \dots, w_n + \tilde{\delta}_n\right)$ is a representation of v for all $\tilde{\delta}_i \in \left[-\frac{1}{5n}, \frac{1}{5n}\right]$, $0 \le i \le n$. With $s = \sum_{i=1}^{n} w_i$ let $\tilde{q} = \frac{1}{s} \cdot \left(q - \frac{2}{5}\right)$ and $\tilde{w}_i = \frac{1}{s} \cdot w_i$ for all $1 \le i \le n-1$. The choice of a suitable α is a bit fiddly ($\alpha = \frac{1}{10ns}$ does work), but its existence is guaranteed by construction. □

Definition 8 *For each weighted majority game v the (normalized) weight polytope $P^{weight}(v)$ is given by $P^{weight}(v) = \big\{w \in \mathbb{R}^n_{\ge 0} \mid \|w\|_1 = 1,\, w(S) \ge w(T) \forall$ min. winning S and all max. losing $T\big\}$.*

By fixing the quota at a suitable value we can directly conclude from Lemma 7:

Corollary 9 *The $n-1$-dimensional volume of $P^{weight}(v)$ is non-zero for each weighted majority game v.*

Lemma 10 *The average weight index of a weighted majority game v is given by*

$$\frac{1}{\int_{P^{weight}(v)} \mathrm{d}\, w} \cdot \left(\int_{P^{weight}(v)} w_1 \,\mathrm{d}\, w, \dots, \int_{P^{weight}(v)} w_n \,\mathrm{d}\, w\right). \tag{2}$$

4.2 The average representation index

As mentioned already in the introduction, one may consider the quota as being part of the weighted representation. To this end we introduce:

Definition 11 *The average representation index of a weighted majority game v is the average of all normalized*[3] *representations of v.*

[3] Taking all representations instead of the normalized ones does not make a difference.

Definition 12 *For each weighted majority game v the (normalized) representation polytope $P^{rep}(v)$ is given by*

$$P^{rep}(v) = \left\{ w \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^{n} w_i = 1, \quad w(S) \geq q \ \forall S \in \mathcal{W}^m, \quad w(T) \leq q \ \forall T \in \mathcal{L}^m \right\}.$$

Using Lemma 3 and Lemma 7 we conclude:

Lemma 13 *The average representation index of a weighted majority game v is given by*

$$\frac{1}{\int_{P^{rep}(v)} \mathrm{d}(q,w)} \cdot \left(\int_{P^{rep}(v)} w_1 \, \mathrm{d}(q,w), \ldots, \int_{P^{rep}(v)} w_n \, \mathrm{d}(q,w) \right). \tag{3}$$

For our example $v = [3; 2, 1, 1]$ we have

$$P^{\text{rep}}(v) = \left\{ (q,w) \in \mathbb{R}^4_{\geq 0} \mid \sum_{i=1}^{3} w_i = 1, w_1 + w_2 \geq q, w_1 + w_3 \geq q, w_1 \leq q, w_2 + w_3 \leq q \right\},$$

and

$$\begin{aligned}
\int_{P^{\text{rep}}(v)} \mathrm{d}(q,w) &= \int_{\frac{1}{2}}^{\frac{2}{3}} \int_{1-q}^{q} \int_{q-w_1}^{1-q} \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q + \int_{\frac{2}{3}}^{1} \int_{2q-1}^{q} \int_{q-w_1}^{1-q} \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q \\
&= \frac{5}{648} + \frac{1}{162} = \frac{1}{72}, \\
\int_{P^{\text{rep}}(v)} w_1 \,\mathrm{d}(q,w) &= \int_{\frac{1}{2}}^{\frac{2}{3}} \int_{1-q}^{q} \int_{q-w_1}^{1-q} w_1 \ \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q + \int_{\frac{2}{3}}^{1} \int_{2q-1}^{q} \int_{q-w_1}^{1-q} w_1 \ \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q \\
&= \frac{31}{7776} + \frac{1}{243} = \frac{7}{864}, \\
\int_{P^{\text{rep}}(v)} w_2 \,\mathrm{d}(q,w) &= \int_{\frac{1}{2}}^{\frac{2}{3}} \int_{1-q}^{q} \int_{q-w_1}^{1-q} w_2 \ \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q + \int_{\frac{2}{3}}^{1} \int_{2q-1}^{q} \int_{q-w_1}^{1-q} w_2 \ \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,q \\
&= \frac{29}{15552} + \frac{1}{972} = \frac{5}{1728},
\end{aligned}$$

so that the average representation index is given by $\left(\frac{7}{12}, \frac{5}{24}, \frac{5}{24}\right)$.

4.3 Properties of the new indices

The two newly introduced indices share several of the properties commonly required for a power index. Three of four properties in Definition 4 are satisfied.

Lemma 14 *The average weight and the average representation index are symmetric, positive, and efficient, satisfies strong monotonicity, but do not satisfy the dummy property.*

PROOF. Symmetry, positivity, and efficiency are inherent in the definition of both indices. The violation of the dummy property can be seen in the example of the game $[1; 1, 0]$. □

The later shortcoming can be repaired using a quite general approach.

Lemma 15 *Given a sequence of power indices $g^n : \mathcal{C}_n \to \mathbb{R}^n$ for all $n \in \mathbb{N}$, let $\tilde{g}^n : \mathcal{C}_n \to \mathbb{R}^n$ be defined via $\tilde{g}^n_i(v) = g^m_i(v')$ for all non-dummies i and by $\tilde{g}^n_j(v) = 0$ for all dummies j, where m is the number of non-dummies in v and v' arises from v by dropping the dummies*[4] *All $\tilde{g}^n$ satisfy the dummy property.*

We call $\tilde{g}^n$ the dummy-revealing version of a given sequence of power indices g^n. For the computation of the dummy-revealing version we just have to compute the dummy reduced game v' and its corresponding power distribution.

The computations from Lemma 10 and Lemma 13 can easily be performed using the software package `LattE` (Baldoni *et al.*, 2014).

game	av. weight	av. rep.	game	av. weight	av. rep.
$[1;1]$	(1)	(1)	$[3;2,1,1,0]$	$\left(\frac{67}{120},\frac{47}{240},\frac{47}{240},\frac{1}{20}\right)$	$\left(\frac{41}{75},\frac{31}{150},\frac{31}{150},\frac{1}{25}\right)$
$[1;1,0]$	$\left(\frac{3}{4},\frac{1}{4}\right)$	$\left(\frac{5}{6},\frac{1}{6}\right)$	$[1;1,1,1,1]$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$
$[1;1,1]$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$[2;1,1,1,1]$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$
$[2;1,1]$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$\left(\frac{1}{2},\frac{1}{2}\right)$	$[3;1,1,1,1]$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$
$[1;1,0,0]$	$\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$	$\left(\frac{3}{4},\frac{1}{8},\frac{1}{8}\right)$	$[4;1,1,1,1]$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$	$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$
$[1;1,1,0]$	$\left(\frac{4}{9},\frac{4}{9},\frac{1}{9}\right)$	$\left(\frac{11}{24},\frac{11}{24},\frac{1}{12}\right)$	$[4;2,1,1,1]$	$\left(\frac{23}{48},\frac{25}{144},\frac{25}{144},\frac{25}{144}\right)$	$\left(\frac{139}{300},\frac{161}{900},\frac{161}{900},\frac{161}{900}\right)$
$[2;1,1,0]$	$\left(\frac{4}{9},\frac{4}{9},\frac{1}{9}\right)$	$\left(\frac{11}{24},\frac{11}{24},\frac{1}{12}\right)$	$[3;2,1,1,1]$	$\left(\frac{7}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16}\right)$	$\left(\frac{43}{100},\frac{19}{100},\frac{19}{100},\frac{19}{100}\right)$
$[1;1,1,1]$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$[2;2,1,1,1]$	$\left(\frac{23}{48},\frac{25}{144},\frac{25}{144},\frac{25}{144}\right)$	$\left(\frac{139}{300},\frac{161}{900},\frac{161}{900},\frac{161}{900}\right)$
$[2;1,1,1]$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$[3;2,2,1,1]$	$\left(\frac{83}{240},\frac{83}{240},\frac{37}{240},\frac{37}{240}\right)$	$\left(\frac{103}{300},\frac{103}{300},\frac{47}{300},\frac{47}{300}\right)$
$[3;1,1,1]$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$	$[4;2,2,1,1]$	$\left(\frac{83}{240},\frac{83}{240},\frac{37}{240},\frac{37}{240}\right)$	$\left(\frac{103}{300},\frac{103}{300},\frac{47}{300},\frac{47}{300}\right)$
$[2;2,1,1]$	$\left(\frac{11}{18},\frac{7}{36},\frac{7}{36}\right)$	$\left(\frac{7}{12},\frac{5}{24},\frac{5}{24}\right)$	$[5;2,2,1,1]$	$\left(\frac{19}{48},\frac{19}{48},\frac{5}{48},\frac{5}{48}\right)$	$\left(\frac{23}{60},\frac{23}{60},\frac{7}{60},\frac{7}{60}\right)$
$[3;2,1,1]$	$\left(\frac{11}{18},\frac{7}{36},\frac{7}{36}\right)$	$\left(\frac{7}{12},\frac{5}{24},\frac{5}{24}\right)$	$[2;2,2,1,1]$	$\left(\frac{19}{48},\frac{19}{48},\frac{5}{48},\frac{5}{48}\right)$	$\left(\frac{23}{60},\frac{23}{60},\frac{7}{60},\frac{7}{60}\right)$
$[1;1,0,0,0]$	$\left(\frac{5}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$	$\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)$	$[4;3,1,1,1]$	$\left(\frac{3}{5},\frac{2}{15},\frac{2}{15},\frac{2}{15}\right)$	$\left(\frac{29}{50},\frac{7}{50},\frac{7}{50},\frac{7}{50}\right)$
$[1;1,1,0,0]$	$\left(\frac{5}{12},\frac{5}{12},\frac{1}{12},\frac{1}{12}\right)$	$\left(\frac{13}{30},\frac{13}{30},\frac{1}{15},\frac{1}{15}\right)$	$[3;3,1,1,1]$	$\left(\frac{3}{5},\frac{2}{15},\frac{2}{15},\frac{2}{15}\right)$	$\left(\frac{29}{50},\frac{7}{50},\frac{7}{50},\frac{7}{50}\right)$
$[2;1,1,0,0]$	$\left(\frac{5}{12},\frac{5}{12},\frac{1}{12},\frac{1}{12}\right)$	$\left(\frac{13}{30},\frac{13}{30},\frac{1}{15},\frac{1}{15}\right)$	$[3;3,2,1,1]$	$\left(\frac{449}{840},\frac{227}{840},\frac{41}{420},\frac{41}{420}\right)$	$\left(\frac{77}{150},\frac{41}{150},\frac{8}{75},\frac{8}{75}\right)$
$[1;1,1,1,0]$	$\left(\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{1}{16}\right)$	$\left(\frac{19}{60},\frac{19}{60},\frac{19}{60},\frac{1}{20}\right)$	$[5;3,2,1,1]$	$\left(\frac{449}{840},\frac{227}{840},\frac{41}{420},\frac{41}{420}\right)$	$\left(\frac{77}{150},\frac{41}{150},\frac{8}{75},\frac{8}{75}\right)$
$[2;1,1,1,0]$	$\left(\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{1}{16}\right)$	$\left(\frac{19}{60},\frac{19}{60},\frac{19}{60},\frac{1}{20}\right)$	$[4;3,2,2,1]$	$\left(\frac{193}{480},\frac{31}{120},\frac{31}{120},\frac{13}{160}\right)$	$\left(\frac{119}{300},\frac{77}{300},\frac{77}{300},\frac{9}{100}\right)$
$[3;1,1,1,0]$	$\left(\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{1}{16}\right)$	$\left(\frac{19}{60},\frac{19}{60},\frac{19}{60},\frac{1}{20}\right)$	$[5;3,2,2,1]$	$\left(\frac{193}{480},\frac{31}{120},\frac{31}{120},\frac{13}{160}\right)$	$\left(\frac{119}{300},\frac{77}{300},\frac{77}{300},\frac{9}{100}\right)$
$[2;2,1,1,0]$	$\left(\frac{67}{120},\frac{47}{240},\frac{47}{240},\frac{1}{20}\right)$	$\left(\frac{41}{75},\frac{31}{150},\frac{31}{150},\frac{1}{25}\right)$			

Table 1: The average weight and average representation index for small games.

Table 1 lists the average weight representation and the average representation index for all weighted majority games with up to four voters. We observe that the

[4]Given a weighted majority game $v : 2^N \to \{0,1\}$ with $S = \{i \in N \mid i \text{ is dummy}\}$, we define the dummy reduced game $v' : 2^{N\backslash S} \to \{0,1\}$ via $v'(T) = v(T)$ for all $T \subseteq N\backslash S$.

so-called dual games obtain the same average weight and average representation index, which can indeed be easily proved.

5 Conclusion and ideas for further research

We have shown how to construct power indices that respect proportionality between power and weight from average representations of a game. By restricting the polytope implied by the set of minimal winning and maximal losing coalitions, we can obtain a representation that is dummy-revealing. The resulting restricted average representation is a coherent measure of power.

The above modification suggests that we can endow the indices with qualities by tailoring the polytope. Indeed, the average representation can be tailored to fulfill additional properties that may lead to other indices. For example, restrictions based on the equivalence classes of voters defined by the Isbell desirability relation lead to a power index that ascribes equal power to all members of an equivalence class. This resulting index is strictly monotonic in voting weight.

The average representations themselves may have other uses, too. They conveniently summarize the set of admissible representations of a weighted majority game into a unique representation, which can then be compared to power distributions of the classical power indices. The average representations can be used to represent a game in simulation studies, much as we use the minimal sum representation to list games in our tables. While each majority game has an infinite number of representations, the number of possible partitions of all games with a given number of voters is finite. This allows us to distinguish between games according to their partitions in equivalence classes, and thus obtain a finite set of games for a comparison between power indices.

We conclude the paper with a remark on integer weights. A normalization of voting weights is unreasonable if they are to represent the number of shares of a corporation, or the number of members of a political party. In these cases, we require the weights to be integers. However, there is still an interpretation of our indices in these cases, as the following convergence result suggests.

Let us return to the initial example in the introduction and consider the weighted majority game $v = [2; 1, 1, 1]$. We have said that 1176 integer weight vectors are feasible for v with a sum of weights 100. If we average those representation, we obtain $\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$ or a relative distribution of $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which is no surprise due to the inherent symmetry. Things get a bit more interesting if one considers the weighted majority game $v = [3; 2, 1, 1]$. For a weight sum of 100, we have 1601 different weight vectors, and the averaged relative weight distribution is given by $(0.608832, 0.195584, 0.195584)$. For a weight sum of 1000, we obtain 166001 different weight vectors and $(0.610888, 0.194556, 0.194556)$. For a weight sum of 10000, we obtain 16660001 different weight vectors and $(0.611089, 0.194456, 0.194456)$. For a weight sum of 100000 we obtain 1666600001 different weight vectors and $(0.611109, 0.194446, 0.194446)$. The averaged relative weight distribution seems to converge to $\left(\frac{11}{18}, \frac{7}{36}, \frac{7}{36}\right)$, which is the average weight index. This can be rigorously proven by numerically approximating the integrals of the definition of the average weight index using grid points only, and considering the limit of increasingly finer equally distributed grids. The same is true if an integer-valued quota is taken into account. At the limit, we would end up with the average representation index. Furthermore, the dummy revealing property can be transferred in this sense.

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