Exploring the dynamics of business survey data using Markov models

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Abstract

Business tendency surveys are widely used for monitoring economic activity. They provide timely feedback on the current business conditions and outlook. We identify the unobserved macroeconomic factors behind the distribution of quarterly responses by Austrian firms on the questions concerning the current business climate and production. The aggregate models use a regime-switching matrix to identify two macroeconomic regimes: upturn and downturn. The micro-founded models envision dependent responses by the firms, so that a favorable or an adverse unobserved common macroeconomic factor increases the frequency of optimistic or pessimistic responses, with the corresponding conditional transition probabilities defined using a coupling scheme. Extensions address the sector dimension and introduce dynamic common tendencies modeled with a hidden Markov chain.

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1 Introduction

Business tendency surveys are an important tool for monitoring the current economic activity and producer sentiment. All industrialized countries and most developing countries conduct regular business surveys by asking the managers of a representative sample of firms about the current state of their business and their expectations for the near future. The Organisation for Economic Co-operation and Development (OECD) and the European Commission (EC) encourage the use of a standardized system of business tendency surveys, thus fostering comparability of results between economies and their constituent sectors (OECD 2003; EC 2014). For the most part, such surveys contain qualitative assessments and expectations. This information typically leads the official economic statistics by a significant number of months, providing timely information on the business cycle and its turning points. Suitably aggregated, the survey data provide valuable information and can be used for constructing more complex composite indicators. Such indicators belong to the standard toolbox of cyclical analysts and forecasters.

The data selected for this study come from the business tendency survey (Konjunkturtest) of the Austrian Institute of Economic Research (WIFO), a monthly survey of a representative sample of Austrian firms. They cover the current business situation and production level in the manufacturing and construction sectors between 1990 and 2017. The study focuses on the patterns of transition in survey responses, with the aim of disentangling their aggregate and sector-specific determinants. The assignment of firms to sectors follows the Statistical Classification of Economic Activities in the European Community (NACE) at the 3-digit level. To reduce the number of sectors, we aggregate the firms in the manufacturing sector using the Main Industrial Groupings (MIG) taxonomy of the Statistical Office of the European Union (Eurostat). The survey is conducted on a monthly basis, but some questions, for example regarding the current business situation, are asked once every quarter. We use quarterly data for estimation.

We explore the survey data using a finite mixture of multinomial distributions – a rich class of latent variable models, in which unobservable variables are assumed to influence the observed response variables. The aggregate and sector-specific conditions are the prime candidates for common unobserved latent variables that affect all firms, or just the firms belonging to a sector of an economy. The inclusion of unobserved variables can be used to account for heterogeneity in the sample, represent measurement errors or group the observations according to unobservable characteristics. Finite mixture latent variable models are well-suited for studying categorical data, in which the response is ordered, for example, from optimistic to neutral and pessimistic. Being versatile and flexible, the use of latent variables finds many applications in different disciplines.[1] Examples include factor models, generalized linear mixed models, finite mixture models, state-space models of longitudinal and panel data and latent Markov models. The estimation techniques may include maximum-likelihood, Bayesian methods or machine learning.

The models estimated in this study assume that firms can be grouped according to their response to a given question at a certain point of time. This constitutes the current assessment of the firm. A transition in the current assessment is defined as the change in the response of a

[1] For example, see the survey by Skrondal and Rabe-Hesketh (2007) and the special issue edited by Alfó and Bartolucci (2015).

firm to the same question between two consecutive sampling dates. First, we use aggregate data to estimate a static and a dynamic version of a model in which the macroeconomic conditions are assumed to be common to all firms. Two regimes of the economy are endogenously identified: downturn and upturn. The dynamic version employs a regime-switching matrix. A more subtle micro-founded model treats outcomes in each group as influenced by a group-specific binary variable representing macroeconomic conditions common to the group, which induces dependence between the responses in the group. The first extension of the micro-founded model introduces a sector dimension in modeling the strength of this dependence. The specific parametrization estimated here was adopted from the credit-risk model by Wozabal and Hochreiter (2012). Being categorical and ordered, credit ratings are similar to survey responses. The main difference between credit rating and survey responses is the presence of a default as an absorbing state, the occurrence of which removes the subject from the sample. There is no such state in a firm survey. The second extension employs a hidden Markov chain to model the dynamics of the unobserved common tendencies.

Following the introduction, Section 2 provides a brief history of the WIFO-Konjunkturtest – the source of data for the present analysis – and describes the sample of firms and the questions used to define the transitions. The counts of particular answers to these questions comprise the estimation sample. In Section 3 we formulate and test aggregate models, static and dynamic. The simplest macro-founded model is introduced in Section 4. It is static and does not account for the sector affiliation of the firm. A static sector micro-founded model is discussed in Section 5, whereas Section 6 describes its dynamic generalization. The final section offers concluding remarks. Further details on the estimators considered in the paper can be found in an appendix.

2 The data

The history of business surveys conducted by WIFO dates back to 1954. First WIFO surveys were inspired by the surveys conducted at the Institute for Economic Research (ifo) in Munich, an institution that has been at the forefront of business cycle research in Germany. Since then, the questionnaire, the periodicity and the size of the sample have changed a number of times. In 1996, it became a part of the Joint Harmonised EU Programme of Business and Consumer Surveys of the European Commission's Directorate General for Economic and Financial Affairs (the "EU Programme" see EC 2014). The foremost aim of the program was to coordinate economic surveys in the EU. Since then, a quarterly survey has been carried out in January, April, July and October.

The responses by the firms can be summarized in various ways, but the typical usage involves constructing indicators in the form of balances, in which the percentage of pessimistic answers to a question is subtracted from the percentage of optimistic answers to the same question. For example, having 30 of 100 respondents expecting production to increase, 50 expecting production to remain constant and 20 expecting a decline, would yield a balance of plus 10 percentage points. The plus indicates that the optimists are in the majority. Although more sophisticated reporting methods exist, the presentation in the form of unweighted balances is common.[2] In general, let f_1 be the frequency of the optimistic responses and f_3 be the frequency of the pessimistic responses.

[2] See, for example, the EU Programme, ifo-Konjunkturtest and KOF Business Tendency Surveys.

Then, the balance, in basic points, is given by

$$(f_1 - f_3) \cdot 100. \tag{1}$$

A negative balance means that the number of pessimists exceeds that of optimists. The idea of reporting balances is attributed to Anderson (1951), who showed that, under certain conditions, changes in the fraction of three-way responses (better-same-worse) can be related to the growth of an underlying economic variable.[3] We follow the convention of using balances in reporting the results.

Numerous empirical studies have addressed the forecasting performance of balance indicators, attesting the usefulness of business tendency surveys as a source of information on the business cycle. In particular, see Hölzl and Schwarz (2014) for a study regarding Austria, Cesaroni (2011) for Italy and Knetsch (2005) for Germany. However, the use of balance indicators implicitly assumes that the firms are homogenous, or that the relevant heterogeneity between the firms nets out in the aggregate. The approach is thus in danger of losing important aspects of firm-specific heterogeneity that might be relevant at the aggregate level. A study of the business cycle by Müller and Köberl (2007) shows how different microeconomic states at the firm level can be compatible with the same macroeconomic aggregate state, a view that has been expressed earlier, for example, by Caballero and Engel (2003) in the context of aggregate investment.

To put the survey size into perspective, around 1,600 Austrian firms with a total of more than 200,000 employees participate in the WIFO-Konjunkturtest on a voluntary basis each month. Of these, around 38 percent are accounted for by the manufacturing sector and 17 percent by the construction sector covered in the present study. Since the number of participants is central to the quality of the indicators determined, WIFO is keen on maintaining a high number of participants by attracting new firms to the survey, resulting in an unbalanced sample, with firms entering and leaving the pool.

Table 1: Number of firms in sectors by five-year periods.

MIG	1991-1995	1996-2000	2001-2005	2006-2010	2011-2015	2016-2017
Intermediate goods (IG)	631	682	616	460	388	291
Capital goods (CG)	65	75	105	74	70	77
Consumer durables (CD)	299	329	286	166	117	72
Consumer non-durables (CP)	245	289	297	261	228	190
Construction (CO)	492	560	598	528	514	564
Total	1732	1935	1902	1489	1317	1194

Turning to the actual sample used in the estimation, the raw survey data spans the interval of 107 quarters between the first quarter of 1991 and the third quarter of 2017. The first time instant is absorbed when transforming the data in transitions between two consecutive quarters, leading to 106 steps along the time axis. Data prior to 1991 are incomparable with more recent vintages. The sector dimension is given by the four categories of the MIG classification: intermediate goods, capital goods, consumer durables and consumer non-durables. The fifth sector in the MIG, the energy sector, is not covered in the survey; we include construction

[3] See the "quantification problem of the economic test" in Geil and Zimmermann (1996).

instead. We label these industries as IG, CG, CD, CP, CO. Table 1 summarizes the number of firms by sector and five-year periods. The sample contains 9569 firms that have responded to the survey at least once between 1991 and 2017. The total in the last row indicates a falling trend in the number of participating manufacturing and construction firms. Construction firms and the producers of intermediate goods provide the most numerous cohorts, the least numerous cohort being the producers of capital goods.

Most of the questions posed in the questionnaire are qualitative. The simplicity of the survey design keeps the burden on participating companies to a minimum. The following two questions were chosen for the collation of transition counts that comprise the estimation sample:

1. The business situation currently is:
   a) better than usual; b) satisfactory; c) worse than usual.

2. Production in the past three months has:
   a) decreased; b) remained unchanged; c) increased.

We refer to the three replies as optimistic (op), neutral (nu) and pessimistic (pe). The first question is more general than the second, because business situation may refer to a wide range of factors, such as profitability, business orders, indebtedness, or fiscal issues. The question on the level of production is more specific and can be related to economic statistics published by a national statistical office. When suitably transformed, the dynamics of aggregate counts for production can be compared to those of the seasonally-adjusted index of production for the manufacturing sector published by Statistics Austria on a monthly basis, which, however, does not include the output of the construction sector.

Having collected answers to the above questions, a matrix of transition counts over a given period of time can be collated. The number of conceivable transition types is $3^2 = 9$ (3 replies). Further differentiating by sectors yields the dimension $5 \cdot 3^2 = 45$ (5 sectors, 3 replies). Taking into account the time dimension spanning 106 quarters yields $106 \cdot 45 = 4770$ as the total number of distinct counts. Note that some of the cells may contain zeroes. This would be the case if the respective types of transitions were not observed in the sample.

3 Aggregate models

3.1 Business situation

We first estimate Markovian transition matrices based on the aggregate counts of the responses to the question on the business situation. In the formulas, the three possible responses – better than usual, satisfactory and worse than usual – will be numbered as 1, 2, 3, with the preference order being $1 \succ 2 \succ 3$. The corresponding labels optimistic (op), neutral (nu) and pessimistic (pe) will be used to label the margins of transition and correlation matrices.

Let T be the total number of quarters; in our sample, $T = 106$. For each quarter t, we consider the firms who participated in the survey in two consecutive quarters t and $t + 1$. Denote by $n_{i,j}(t)$ the number of firms that replied i at t and j at $t + 1$. For example, $n_{1,2}(t)$ is the number of firms that turned from optimistic to neutral at time t. Then, the frequencies

$$P_{i,j} = \frac{\sum_{t=1}^{T} n_{i,j}(t)}{\sum_{t=1}^{T}[n_{i,1}(t) + n_{i,2}(t) + n_{i,3}(t)]}, \quad i, j = 1, 2, 3.$$

can be interpreted as transition probabilities of the corresponding Markov chain. Set $\vec{f} = (f_1, f_2, f_3)$ for the distribution of responses,

$$f_i = \frac{\sum_{t=1}^{T}[n_{i,1}(t) + n_{i,2}(t) + n_{i,3}(t)] + n_i(T+1)}{\sum_{t=1}^{T} n(t) + n(T+1)}, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} n(t) &= \sum_{i=1}^{3}\sum_{j=1}^{3} n_{i,j}(t), \\ n_i(T+1) &= n_{1,i}(T) + n_{2,i}(T) + n_{3,i}(T), \\ n(T+1) &= n_1(T+1) + n_2(T+1) + n_3(T+1). \end{aligned}$$

The balance, or the surplus of optimists over pessimists, is given by $(f_1 - f_3) \cdot 100$.

For our sample, we obtain the following transition matrix and vector of averages:

$$P = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} \quad op & \quad nu & \quad pe \end{array} \\ \begin{pmatrix} 0.5794 & 0.3818 & 0.0389 \\ 0.1164 & 0.7297 & 0.1538 \\ 0.0234 & 0.3252 & 0.6514 \end{pmatrix} \end{array}, \ \vec{f} = (0.1634, 0.5635, 0.2731).$$

The balance of $(0.1634 - 0.2731) \cdot 100 = -10.97$ shows that the pessimists are, on average, in the majority. We can validate this outcome by comparing the steady-state distribution of the Markov chain to the observed averages. The matrix P, being irreducible and aperiodic, has a unique steady-state distribution $\vec{\pi} = (0.1705, 0.5623, 0.2672)$. Since $\vec{\pi} = \lim_{t\to\infty} \vec{\pi}^{(0)} P^t$ for every initial distribution $\vec{\pi}^{(0)}$, the vector $\vec{\pi}$ can be regarded as a long-run limit for the frequencies $\vec{f}$. It suggests a slightly less pessimistic balance -9.67 in the long run.

3.2 Identifying two regimes

Instead of a single transition matrix P, we can try identifying two transition matrices: one for *downturns*, P^D, and one for *upturns*, P^U. Assume that the opinion formation process is governed by a mixture of P^D and P^U with the weights p and $1 - p$, where p is the probability of an upturn quarter and $(1 - p)$ is the probability of a downturn quarter. We would like the model to identify the matrices endogenously, returning P and $p = 1$ if the two regimes cannot be distinguished. Entries of the regime-specific matrices must satisfy the following inequalities:

$$P^D_{i,j} \geq P^U_{i,j} \text{ if } j > i \text{ and } P^D_{i,j} \leq P^U_{i,j} \text{ if } j < i. \tag{2}$$

In view of the response ranking, the inequalities imply that transition probabilities to better states are higher in upturns, whereas the transition probabilities to worse states are higher in downturns. We can impose a minimum threshold for this variation by requiring

$$\begin{aligned} &P^D_{1,2} - P^U_{1,2} + P^D_{1,3} - P^U_{1,3} \geq \epsilon_1, \\ &P^U_{2,1} - P^D_{2,1} + P^D_{2,3} - P^U_{2,3} \geq \epsilon_2, \\ &P^U_{3,1} - P^D_{3,1} + P^U_{3,2} - P^D_{3,2} \geq \epsilon_3, \end{aligned} \tag{3}$$

were ϵ_i are some non-negative numbers. If all $\epsilon_i = 0$, inequalities (3) follow from inequalities (2). Since we do not have a prior assumption about the firms who retain their attitudes between two consecutive quarters, the diagonal probabilities $P_{i,i}$ are not explicitly modified. They still can be affected by the variation of the off-diagonal probabilities, however, since the row entries must sum to unity.

The above model was estimated for the following three threshold values 0, 0.05 and 0.1, assuming that $\epsilon_1 = \epsilon_2 = \epsilon_3$. Since the choice of the threshold value had a negligible effect on the estimated transition matrices, we present the results for $\epsilon_i = 0$, $i = 1, 2, 3$, only. We have,

$$P^U = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.6279 & 0.3452 & 0.0269 \\ 0.1391 & 0.7457 & 0.1151 \\ 0.0293 & 0.3844 & 0.5862 \end{pmatrix} \end{array}, \quad P^D = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5043 & 0.4383 & 0.0574 \\ 0.0902 & 0.7110 & 0.1988 \\ 0.0185 & 0.2770 & 0.7046 \end{pmatrix} \end{array}, \quad p = 0.5414.$$

The above estimates suggest that 54 percent of all quarters were upturns. This value is supported by the fact that the seasonally-adjusted index of production for the manufacturing sector has accelerated in 52 percent of all quarters contained in the sample, relative to the same quarter in the previous year. The two figures agree surprisingly well, despite the fact that the production index does not include construction that is included in our sample. The comparable figure for the aggregate value-added (real GDP) equals 46 percent, with the larger difference being attributed to a moderate share of manufacturing in the economy and the conceptual difference between production and the value added.

Table 2 shows the variation of $P_{i,i}$, i.e. $(P^U_{i,i}/P_{i,i} - 1) \cdot 100$ and $(P^D_{i,i}/P_{i,i} - 1) \cdot 100$. Definite opinions are more strongly influenced by macroeconomic factors than neutral ones. The optimistic and neutral responses adjust opposite to the direction of adjustment of pessimistic responses.

Table 2: Business situation. Variation of $P_{m,m}$.

	Firm response m		
	op	*nu*	*pe*
Upturn	8.37	2.19	-10.01
Downturn	-12.96	-2.56	8.17

The steady-state distributions for an upturn and a downturn regime are:

$$\vec{\pi}^U = (0.2338, 0.5875, 0.1787), \quad \vec{\pi}^D = (0.1085, 0.5203, 0.3712).$$

The corresponding balances equal 5.5 and -16.27. Although these values are never attained, they represent bounds for what can be observed in the long run. The balances calculated according to $\vec{f}P^U$ correspond to what can plausibly be expected following one quarter of upturn. In fact, having no particular indication for choosing the initial state, $\vec{f}$ has to be taken as a typical state. After one quarter of upturn, this distribution transforms into $\vec{f}P^U$. In the same way, after k subsequent upturn quarters, multiplication by P^U has to be performed k times. As $k \to \infty$, the limit will be $\vec{\pi}^U$. Similarly, during downturn quarters, the transition matrix P^D governs the dynamics of shares. Table 3 illustrates the evolution of balances.

Table 3: Business situation. Balances in two regimes.

	Number of subsequent quarters			
	1	2	3	4
Upturn	-4.03	0.00	2.31	3.66
Downturn	-17.55	-21.30	-23.44	-24.66

3.3 Regime switching

A richer dynamic of the opinion-formation process can be achieved by allowing a downturn quarter to follow an upturn quarter with probability α, and a downturn quarter to follow a downturn quarter with probability β, so that the following regime-switching matrix

$$\mathcal{P} = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} U & \quad D \end{array} \\ \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} \end{array}$$

governs the evolution of the phases of a business cycle.

Similar to the static model, the results are very close for the three threshold values. The following estimates were obtained for $\epsilon_i = 0$, $i = 1, 2, 3$:

$$P^U = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.6278 & 0.3452 & 0.0270 \\ 0.1391 & 0.7458 & 0.1151 \\ 0.0292 & 0.3847 & 0.5861 \end{pmatrix} \end{array}, \quad P^D = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5048 & 0.4381 & 0.0572 \\ 0.0902 & 0.7108 & 0.1989 \\ 0.0184 & 0.2769 & 0.7047 \end{pmatrix} \end{array},$$

$$\mathcal{P} = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \begin{pmatrix} 0.5932 & 0.4068 \\ 0.4068 & 0.5932 \end{pmatrix} \end{array}, \quad p = 0.5373.$$

The transition matrices and the probability p are very close to their static counterparts, which lends credibility to the static model with two regimes. Since the distribution $(0.5373, 0.4627)$ deviates from the steady-state distribution $(0.5, 0.5)$ of $\mathcal{P}$ by only 0.0303, we conclude that the dynamic of responses is not informative of the underlying macroeconomic dynamic.

There is an apparent ambiguity in the results discussed so far. Upturn quarters seem to be 4 percent more frequent than downturn quarters, yet the firms are, on average, pessimistic about their current business situation, as suggested by the negative balance of -10.97. This may be related to the generality of the question and suggest a pessimism bias in the responses (Bachmann and Elstner 2015). With this in mind, let us turn to the question on the level of production in the past three months.

3.4 Production

The wording on the question on production implies that a transition between answer i to answer j should be interpreted as the change in the assessment of the level of production in the previous quarter. For this question, we obtain the following transition matrix:

$$P = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4957 & 0.3986 & 0.1057 \\ 0.1772 & 0.6873 & 0.1355 \\ 0.1018 & 0.4356 & 0.4626 \end{pmatrix} \end{array}, \ \vec{f} = (0.2350, 0.5711, 0.1939),$$

$$\vec{\pi} = (0.2389, 0.5703, 0.1908).$$

The balance based on the above averages equals 4.11, whereas the long-run balance based on the steady-state distribution equals 4.81. For the static setting, the transition matrices and upturn probabilities coincide for all three threshold levels. The corresponding steady-state distributions are

$$P^U = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5476 & 0.3779 & 0.0745 \\ 0.2068 & 0.6932 & 0.1000 \\ 0.1305 & 0.4748 & 0.3947 \end{pmatrix} \end{array}, \ \vec{\pi}^U = (0.2986, 0.5704, 0.1310).$$

$$P^D = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4070 & 0.4339 & 0.1591 \\ 0.1360 & 0.6792 & 0.1849 \\ 0.0743 & 0.3980 & 0.5277 \end{pmatrix} \end{array}, \ \vec{\pi}^D = (0.1633, 0.5618, 0.2749), \ p = 0.5887.$$

The variation of $P_{i,i}$ is shown in Table 4. The variation for the definite opinions exceeds

Table 4: Production level. Variation of $P_{m,m}$.

	Firm response m		
	op	*nu*	*pe*
Upturn	10.47	0.86	-14.68
Downturn	-17.89	-0.01	14.07

its counterparts in Table 2. This increased articulation of the definite opinions can be due to the fact that the question on production is more specific than the question on the business conditions, and because the immediate past may be more certain than the present.

It appears that almost 59 percent of all quarters were upturns. This value is sufficiently close to the percentage of quarters in which the seasonally-adjusted index of production for the manufacturing sector has accelerated relative to the same quarter in the previous year, which lends credence to the business cycle interpretation of the common factor. This optimistic conclusion is consistent with the balance for the steady-state distribution $\vec{\pi}^U$ of P^U is 16.76; its counterpart for P^D equals -11.16.

Balances emerging from the two regimes (starting from the average $\vec{f}$) are given in Table 5.

Table 5: Production. Balances in two regimes.

	Number of subsequent quarters.			
	1	2	3	4
Upturn	12.10	15.04	16.13	16.53
Downturn	-5.76	-9.24	-10.48	-10.92

Turning to the dynamic setting, we see that all three thresholds imply identical estimates:

$$P^U = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5476 & 0.3779 & 0.0745 \\ 0.2067 & 0.6933 & 0.1000 \\ 0.1304 & 0.4747 & 0.3949 \end{pmatrix} \end{array}, \quad P^D = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4069 & 0.4339 & 0.1592 \\ 0.1360 & 0.6791 & 0.1850 \\ 0.0744 & 0.3980 & 0.5276 \end{pmatrix} \end{array},$$

$$\mathcal{P} = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \begin{pmatrix} 0.7733 & 0.2267 \\ 0.3146 & 0.6854 \end{pmatrix} \end{array}, \ p = 1.0000.$$

The dynamic model indicates that upturns are more frequent than downturns. The transition probabilities P^U and P^D are very similar to their static counterparts, while the upturn probability differs considerably. To understand this result, observe that the steady-state distribution of the regime-switching matrix is $(0.5812, 0.4188)$. The static model accounts for this distribution implicitly, as probabilistic weights which bring the two transition matrices into an equilibrium with the flow of transition counts. In the dynamic setting, the estimate of the mixing probability $p = 1.0000$ implies that the model identifies the initial state as an upturn.

To conclude the discussion of the dynamic aggregate model, we would like to mention that a dynamic model allows us to analyze multi-quarter scenarios. Denoting upturns and downturns by the letters U and D, the sequence $UDDU$ would correspond to a four-quarter period that begins with an upturn quarter that is followed by two subsequent downturn quarters and that ends with an upturn quarter. The probabilities of the multi-quarter scenarios listed in Table 6 and 7 are evaluated using the formula of total probability and the regime-switching matrix $\mathcal{P}$. Note that the order of upturns or downturns along with their numbers affect the resulting balances. Recall that the balance is the surplus of optimists over pessimists. The balances are positive for any sequence that starts with an upturn and ends with an upturn. On the contrary, any sequence that starts with an upturn and ends with an downturn yields a negative balance.

Table 6: Production. Two and three quarter balances.

	Scenario					
	UU	UD	UUU	UUD	UDU	UDD
Probability	0.7733	0.2267	0.5980	0.1753	0.1753	0.0514
Balance	15.03	-2.94	16.11	-1.92	9.49	-8.24

Table 7: Production. Four quarter balances.

	Scenario							
	$UUUU$	$UUUD$	$UUDU$	$UUDD$	$UDUU$	$UDUD$	$UDDU$	$UDDD$
Probability	0.4624	0.1356	0.1356	0.040	0.1356	0.040	0.040	0.0118
Balance	16.50	-1.56	9.86	-7.89	14.07	-3.84	7.52	-10.13

4 A micro-founded model

The micro-founded model moves from an aggregate perspective on the dynamic of survey responses to explicitly modeling the response of a firm. The key assumption is that the response constitutes a noisy signal about the underlying unobserved macroeconomic conditions. Assume that each firm n responds according to a macroeconomic factor, or idiosyncratically. Which of these two possibilities occurs is modeled as a binary random variable that controls how informative the response of a firm is.

The *idiosyncratic* component of firm n that responds i is determined by the random variables ξ_n taking the values 1, 2 and 3, with probabilities $P_{i,1}$, $P_{i,2}$, $P_{i,3}$. These values correspond to the optimistic, neutral and pessimistic responses. Idiosyncratic is synonymous with independent in time and across the firms. Here and below we simplify the notation by dropping the time index.

The unobserved *macroeconomic factors* can be *favorable* or *adverse*, encoded by 1 and 0. For three possible responses to a survey question, a binary vector $\vec{\chi} = (\chi_1, \chi_2, \chi_3)$ denotes a *macroeconomic scenario*. Let its i-th coordinate χ_i encode the macroeconomic factor affecting the firms that replied i (optimistically, neutrally or pessimistically). There are eight such scenarios in total. For example, $(1, 1, 1)$ corresponds to a macroeconomic scenario favorable to all firms covered by the survey, and $(1, 0, 0)$ corresponds to a scenario favorable to all firms that replied optimistically.

We can now define the following conditional probabilities:

$$
\begin{aligned}
&P_{1,1}(1)=1,\ P_{1,2}(1)=P_{1,3}(1)=0;\\
&P_{1,1}(0)=0,\ P_{1,2}(0)=\frac{P_{1,2}}{P_{1,2}+P_{1,3}},\ P_{1,3}(0)=\frac{P_{1,3}}{P_{1,2}+P_{1,3}};\\
&P_{2,1}(1)=\frac{P_{2,1}}{P_{2,1}+dP_{2,2}},\ P_{2,2}(1)=\frac{dP_{2,2}}{P_{2,1}+dP_{2,2}},\ P_{2,3}(1)=0;\\
&P_{2,1}(0)=0,\ P_{2,2}(0)=\frac{(1-d)P_{2,2}}{P_{2,3}+(1-d)P_{2,2}},\ P_{2,3}(0)=\frac{P_{2,3}}{P_{2,3}+(1-d)P_{2,2}};\\
&P_{3,1}(1)=\frac{P_{3,1}}{P_{3,1}+P_{3,2}},\ P_{3,2}(1)=\frac{P_{3,2}}{P_{3,1}+P_{3,2}},\ P_{3,3}(1)=0;\\
&P_{3,1}(0)=P_{3,2}(0)=0,\ P_{3,3}(0)=1.
\end{aligned}
$$

The model parameter $d \in [0,1]$ controls the relation between $P_{2,2}(1)$, $P_{2,2}(0)$ and $P_{2,2}$. Depending on whether d is larger or smaller than $P_{2,1}/(1-P_{2,2})$, the probabilities $P_{2,2}(1)$ ($P_{2,2}(0)$) will be larger (smaller) or smaller (larger) than $P_{2,2}$. This is a counterpart to the implicit adjustment mechanism of the diagonal probabilities in the aggregate models.

The indicator random variable δ_n controls how informative the response of a firm is about the underlying macroeconomic factor. The response is idiosyncratic if $\delta_n = 1$. The response is informative of the underlying macroeconomic factor (that is common to all firms that replied in a certain way) if $\delta_n = 0$. Let the randomizing probability q_i be the same for all firms that replied i, or $\mathbb{P}\{\delta_n = 1\} = q_i$. The indicators are independent in time and across the firms.

Having defined an idiosyncratic factor, a scenario comprising macroeconomic factors and a mixing variable, we can now specify the response of a firm ζ_n. Let ζ_n depend on the macroeconomic factor determining the response of firm n. If the current macroeconomic scenario is $\vec{\chi}$, and the firm responds i, then the distribution of ζ_n conditional on the macroeconomic factor is given by $P_{i,1}(\chi_i)$, $P_{i,2}(\chi_i)$, $P_{i,3}(\chi_i)$. Conditioned on the factor, the random variables ζ_n are independent in time and across the firms. Suppressing the superscript t, the *response* of n is represented as $\delta_n\xi_n + (1-\delta_n)\zeta_n$ at every time instant t. Given a macroeconomic scenario, the families of random variables $\{\delta_n\}$, $\{\xi_n\}$ and $\{\zeta_n\}$ are independent in time and across the firms. The responses in each quarter are described by a mixture of multinomial distributions. The corresponding likelihood function is given by:

$$
L(\vec{\rho},\vec{q},d)=\prod_{t=1}^{T}\sum_{k=1}^{8}\rho_k\prod_{i=1}^{3}\prod_{j=1}^{3}[q_iP_{i,j}+(1-q_i)P_{i,j}(\chi_i^{(k)})]^{n_{i,j}(t)}.
$$

We arrange the vectors $\vec{\chi}$ from $\vec{\chi}^{(1)} = (1,1,1)$ to $\vec{\chi}^{(8)} = (0,0,0)$, or in the descending order of the integers they represent in a binary way. The k-th coordinate ρ_k of the 8-dimensional vector $\vec{\rho}$ is the probability of the macroeconomic scenario $\vec{\chi}^{(k)}$. The vectors $\vec{\rho}$ and $\vec{q}$ as well as the constant d are estimated by maximizing $\ln L(\vec{\rho},\vec{q},d)$, subject to the inequality constraints, q_i, ρ_j, $d \in [0,1]$, and the following linear equality constraints:

$$
\rho_1+\rho_2+\rho_3+\rho_4+\rho_5+\rho_6+\rho_7+\rho_8=1,
$$

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 = P_{1,1},\ \rho_1 + \rho_2 + \rho_5 + \rho_6 - dP_{2,2} = P_{2,1},\ \rho_1 + \rho_3 + \rho_5 + \rho_7 = P_{3,1} + P_{3,2}. \quad (4)$$

The first equality ensures that the coordinates of $\vec{\rho}$ constitute a probability distribution. Since $\vec{\chi}^{(1)} = (1,1,1)$, $\vec{\chi}^{(2)} = (1,1,0)$, $\vec{\chi}^{(3)} = (1,0,1)$ and $\vec{\chi}^{(4)} = (1,0,0)$, the first of equalities (4) implies that a macroeconomic scenario favorable to all optimistic firms, or a binary vector representing the macroeconomic scenario with 1 as its first coordinate, occurs with the probability $P_{1,1}$. Since the second coordinate of the binary vectors $\vec{\chi}^{(1)}$, $\vec{\chi}^{(2)}$, $\vec{\chi}^{(5)}$ and $\vec{\chi}^{(6)}$ equals 1, the second equality in (4) implies that a macroeconomic scenario favorable to all neutral firms occurs with the probability $P_{2,1} + dP_{2,2}$. Similarly, the third equality in (4) defines the probability that a macroeconomic scenario is favorable for all pessimistic firms.

Given the above conditional probabilities, the unconditional distribution of ζ_n coincides – by the formula of total probability – with the i-th row of P when n responds i, and the i-th linear equality in (4) occurs. Unconditionally, the response of firm i is governed by the mixture $q_i P_{i,j} + (1 - q_i) P_{i,j} = P_{i,j}$, where q_i is the probability that the signal is idiosyncratic. It is important to emphasize that the linear equalities (4) guarantee the consistency of the conditional distribution and the mixing scheme with the unconditional dynamic defined by P. The consistency of the micro-founded model with the aggregate dynamic is an attractive feature of models based on coupling schemes.

The above dependence pattern among the random variable $\{\zeta_n\}$ is a modification of the coupling scheme suggested by Wozabal and Hochreiter (2012) for the modeling of dependent credit-rating migrations. This is not the only way to introduce dependence among the responses. A stronger correlation among individual responses can be achieved by using the specification in Boreiko et al. (2016). An even stronger dependence pattern can be obtained for an analog of the coupling scheme by Kaniovski and Pflug (2007).

Since results based on the aggregate estimates for the current business situation have been inconclusive, we focus on the level of production in the past three months. For these data, the following vectors $\vec{q}$, $\vec{\rho}$ and constant d have been obtained:

$$\vec{q} = (0.8413, 0.7436, 0.8320);$$

$$\vec{\rho} = (0.3426, 0.1070, 0.0000, 0.0462, 0.1281, 0.0000, 0.0667, 0.3095);\ d = 0.5826.$$

Note that two entries of $\vec{\rho}$ vanish, meaning that two of the eight macroeconomic scenarios have zero probability of occurrence and can be excluded from further analysis. The non-vanishing entries show that 34.26 percent (30.95 percent) of all quarters were favorable (adverse) for all firms, and that 49.58 percent, 57.77 percent and 53.74 percent of quarters were favorable for the optimistic, neutral and pessimistic firms, respectively. These percentages are obtained by adding probabilities of all binary vectors $\vec{\chi}^{(k)}$ having 1 at the corresponding position. Despite all three figures being close to 50 percent, neutral respondents appear to experience favorable macroeconomic conditions more frequently than pessimists and optimists.

The value $d = 0.5826$ implies that the conditional probabilities $P_{2,2}(\chi_2)$ are close to $P_{2,2}$: $P_{2,2}(1)$ is 0.09 percent larger, while $P_{2,2}(0)$ is 1.19 percent smaller. Since none of q_i equals 1, the responses depend on the macroeconomic factor. The dependence on a common factor induces dependence between the responses of the firms, as conveyed by the following correlation matrix:

$$C = \begin{pmatrix} 1.0000 & 0.6609 & 0.3057 \\ 0.6609 & 1.0000 & 0.6507 \\ 0.3057 & 0.6507 & 1.0000 \end{pmatrix}.$$

$C_{i,j}$ is the correlation coefficient between the event of a macroeconomic scenario favorable to the firms responding with i and j. Recall that they are encoded as 1 in the corresponding position of the binary vector representing a scenario. Since the off-diagonal elements differ from 1, the macroeconomic factors affecting different respondents are not identical. Since $C_{1,2} > C_{1,3}$, the neighboring responses 1 and 2 are more strongly dependent than 1 and 3. The neutral opinion correlates equally strongly with the pessimistic and the optimistic opinion, as $C_{2,3} \approx C_{2,1}$.

4.1 Upturns, downturns and the representative firm

The first macroeconomic scenario $(1,1,1)$ represents a clear upturn, with favorable macroeconomic conditions for the optimists, neutrals and pessimists, whereas the scenario $(0,0,0)$ is a clear downturn. We call these scenarios *polar*, and the remaining scenarios *mixed*. A mixed scenario represents a partial upturn or a partial downturn. The presence of mixed scenarios complicates the comparison of the results obtained using the micro-founded models with the estimates for the aggregate models with two regimes. This is because several such scenarios could be subsumed in the two regimes estimated by the aggregate model.

Consider the percentages corresponding to the two polar scenarios. For the macroeconomic scenario $(1,1,1)$, the matrix $P^{(+)}$ formed by $P_{i,j}(1)$ equals

$$\begin{array}{cc} & \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{array}{c} op \\ nu \\ pe \end{array} & \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.3068 & 0.6932 & 0.0000 \\ 0.1894 & 0.8106 & 0.0000 \end{pmatrix} \end{array}.$$

The above estimates cannot be compared with those obtained using an aggregate model, because the randomizing behavior of actual firms in the sample may not match this transition matrix. To make the estimates of the micro-founded model comparable to those of an aggregate model, we need to construct a *representative firm* by averaging over the types of responses according to their frequencies in the sample. This would be a hypothetical firm, whose responses under the macroeconomic scenario $(1,1,1)$ are governed by $Q^{(1)}$. The matrix $Q^{(1)}$, whose entries are given by $Q^{(1)}_{i,j} = q_i P_{i,j} + (1-q_i) P^{(+)}_{i,j}$, governs the responses of a representative firm:

$$\begin{array}{cc} & \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{array}{c} op \\ nu \\ pe \end{array} & \begin{pmatrix} 0.5757 & 0.3354 & 0.0889 \\ 0.2105 & 0.6887 & 0.1008 \\ 0.1165 & 0.4986 & 0.3849 \end{pmatrix} \end{array}.$$

The above transition matrix of a representative firm is conditioned on a macroeconomic scenario indicated by the superscript. It is important to emphasize that conditional transition matrices for any scenario can be expressed in terms of conditional probabilities for the two polar scenarios $(1,1,1)$ and $(0,0,0)$. We therefore only need to consider the conditional transition matrices of the polar scenarios.

For the macroeconomic scenario $(0,0,0)$, the following matrix $P^{(-)}$ obtains

$$\begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} \quad op & \quad\quad nu & \quad\quad pe \end{array} \\ \begin{pmatrix} 0.0000 & 0.7904 & 0.2096 \\ 0.0000 & 0.6792 & 0.3208 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \end{array}.$$

Then, the matrix

$$Q^{(8)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} \quad op & \quad\quad nu & \quad\quad pe \end{array} \\ \begin{pmatrix} 0.4170 & 0.4608 & 0.1222 \\ 0.1317 & 0.6852 & 0.1831 \\ 0.0847 & 0.3624 & 0.5529 \end{pmatrix} \end{array},$$

whose entries are given by $Q^{(8)}_{i,j} = q_i P_{i,j} + (1-q_i) P^{(-)}_{i,j}$, governs the responses of the representative firm under this scenario.

The steady-state distributions of $Q^{(1)}$ and $Q^{(8)}$ are:

$$\vec{\pi}^{(1)} = (0.3115, 0.5529, 0.1356), \quad \vec{\pi}^{(8)} = (0.1662, 0.5593, 0.2745).$$

The values of the respective balances 17.59 and -10.33 are consistent with their counterparts for the static aggregate model.

Table 8: Production. Balances in polar regimes.

	Number of subsequent quarters			
	1	2	3	4
Polar upturn	12.50	15.67	16.87	17.31
Polar downturn	-5.09	-8.60	-9.96	-10.49

Table 9: Production. Balances in mixed regimes.

		Number of subsequent quarters			
Scenario	Probability	1	2	3	4
$(1,1,0)$	0.1070	8.62	10.49	11.28	11.62
$(1,0,0)$	0.0462	-0.58	-2.63	-3.55	-3.97
$(0,1,1)$	0.1281	7.99	9.34	9.78	9.91
$(0,0,1)$	0.0667	-1.21	-2.87	-3.38	-3.54

Table 8 shows the percentages generated by $Q^{(1)}$ and $Q^{(8)}$ if the dynamic starts with $\vec{f}$. These values can be compared to those in Table 5. Despite the implementation of upturns and downturns in the micro-founded model being more restrictive than in the aggregate models, the corresponding percentages are similar. For the four remaining feasible macroeconomic scenarios, 2, 4, 5 and 7, the percentages are presented in Table 9, where feasible means that the corresponding probabilities differ from zero. Here, the probability of a scenario is understood as the probability that a randomly chosen quarter from the period of observation is characterized by the macroeconomic conditions encoded by corresponding binary vector $\vec{\chi}^{(i)}$. The corresponding conditional transition matrices $Q^{(i)}$ consist of the respective rows of $Q^{(1)}$ and $Q^{(8)}$. For example, under the second scenario, 10.70 percent of all quarters are favorable for the optimistic and the neutral firms, while being adverse for the pessimistic firms. Consequently, the first two rows of $Q^{(2)}$ coincide with the first two rows of $Q^{(1)}$, and the third row coincides with the third row in $Q^{(8)}$.

5 A micro-founded model with sector differentiation

The models estimated so far have ignored the sector dimension captured by the MIG classification, which covers the following industries: intermediate goods, capital goods, consumer durables and consumer non-durables. The fifth sector is construction. We label these five industries as IG, CG, CD, CP, CO. A refinement of the above micro-founded model allows characterizing industry-specific effects. The micro-founded model with sector differentiation requires industry-specific response counts $n_{i,j}^{(s)}(t)$, such that $n_{i,j}(t) = n_{i,j}^{(1)}(t) + n_{i,j}^{(2)}(t) + n_{i,j}^{(3)}(t) + n_{i,j}^{(4)} + n_{i,j}^{(5)}(t)$, as input data.

The first step in modeling the sector effect is to consider δ_n with industry-specific probabilities of success. Denote by $q_{i,s}$ entries of the corresponding 3×5 matrix q, i stands for the type of a firm response and s stands for the industry. The estimator given in the appendix is similar to its counterpart for the model without differentiation. The transition counts and the elements of $\vec{q}$ must be split among the industries in the likelihood function. We need more inequality constraints to incorporate industry-specific $q_{i,s}$, but the linear equality constraints remain valid. Recall that $q_{i,s}$ is the probability of an idiosyncratic response that is not informative of the aggregate conditions.

The following q, d and $\vec{\rho}$ were estimated for the question about the production level in the past three months:

$$
q = \begin{array}{c} \\ op \\ nu \\ pe \end{array}
\begin{array}{c} \begin{array}{ccccc} \quad IG \quad & \quad CG \quad & \quad CD \quad & \quad CP \quad & \quad CO \quad \end{array} \\
\begin{pmatrix} 0.8460 & 0.8354 & 0.7498 & 0.8212 & 0.9834 \\ 0.7228 & 0.8562 & 0.8474 & 0.7449 & 0.8022 \\ 0.7811 & 0.8926 & 0.8631 & 0.8660 & 0.9893 \end{pmatrix} \end{array}, \ d = 0.7133;
$$

$$
\vec{\rho} = (0.3656, 0.1301, 0.0000, 0.0000, 0.1718, 0.0000, 0.0000, 0.3325).
$$

The value $d = 0.7133$ implies a noticeable variation of the conditional probabilities $P_{2,2}(\chi_2)$ against their unconditional counterparts: $P_{2,2}(1)$ is 6.87 percent larger, while $P_{2,2}(0)$ is 13.79 percent smaller. Since none of the entries $q_{i,j}$ equals to 1, opinion-formation in all industries indeed depends on macroeconomic factors, and these common factors induce the dependence among the firm responses in the survey.

The probability of a polar upturn (downturn) quarter is 0.3656 (0.3325). These values are similar to those estimated using the model without sector differentiation. The columns of q

resemble the vector $\vec{q}$ reported above. The correlation matrix

$$C = \begin{pmatrix} 1.0000 & 0.6998 & 0.3981 \\ 0.6998 & 1.0000 & 0.7607 \\ 0.3981 & 0.7607 & 1.0000 \end{pmatrix}$$

is also similar to its previously estimated counterpart. In sum, the estimates corresponding to a simplified model and its generalization are very close, which is an argument in favor of both models.

As in the previous case, all industry-specific $Q^{(i,s)}$ are obtained by combining the corresponding rows of $Q^{(1,s)}$ and $Q^{(8,s)}$. These matrices characterize the impact of extreme scenarios of a polar upturn and a polar downturn on the respective industries. Next, we quote them with the respective averages $\vec{f}^{(s)}$ corresponding to the whole sample:

$$Q^{(1,IG)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5734 & 0.3372 & 0.0894 \\ 0.2016 & 0.7005 & 0.0979 \\ 0.1210 & 0.5176 & 0.3614 \end{pmatrix} \end{array}, \quad Q^{(8,IG)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4194 & 0.4589 & 0.1217 \\ 0.1280 & 0.6611 & 0.2109 \\ 0.0796 & 0.3402 & 0.5802 \end{pmatrix} \end{array},$$

$$\vec{f}^{(IG)} = (0.2277, 0.5749, 0.1974);$$

$$Q^{(1,CG)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5787 & 0.3330 & 0.0883 \\ 0.1899 & 0.6941 & 0.1160 \\ 0.1112 & 0.4759 & 0.4129 \end{pmatrix} \end{array}, \quad Q^{(8,CG)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4141 & 0.4631 & 0.1228 \\ 0.1517 & 0.6737 & 0.1746 \\ 0.0909 & 0.3888 & 0.5203 \end{pmatrix} \end{array},$$

$$\vec{f}^{(CG)} = (0.2631, 0.5410, 0.1959);$$

$$Q^{(1,CD)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.6219 & 0.2988 & 0.0793 \\ 0.1906 & 0.6946 & 0.1148 \\ 0.1138 & 0.4869 & 0.3993 \end{pmatrix} \end{array}, \quad Q^{(8,CD)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.3717 & 0.4966 & 0.1317 \\ 0.1501 & 0.6729 & 0.1770 \\ 0.0879 & 0.3759 & 0.5362 \end{pmatrix} \end{array},$$

$$\vec{f}^{(CD)} = (0.2064, 0.5910, 0.2026);$$

$$Q^{(1,CP)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5859 & 0.3273 & 0.0868 \\ 0.1997 & 0.6994 & 0.1009 \\ 0.1136 & 0.4858 & 0.4006 \end{pmatrix} \end{array}, \quad Q^{(8,CP)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4071 & 0.4686 & 0.1243 \\ 0.1320 & 0.6632 & 0.2048 \\ 0.0882 & 0.3772 & 0.5346 \end{pmatrix} \end{array},$$

$$\vec{f}^{(CP)} = (0.2601, 0.5591, 0.1808);$$

$$Q^{(1,CO)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.5041 & 0.3920 & 0.1039 \\ 0.1946 & 0.6967 & 0.1087 \\ 0.1027 & 0.4396 & 0.4577 \end{pmatrix} \end{array}, \quad Q^{(8,CO)} = \begin{array}{c} \\ op \\ nu \\ pe \end{array} \begin{array}{c} \begin{array}{ccc} op & nu & pe \end{array} \\ \begin{pmatrix} 0.4875 & 0.4051 & 0.1074 \\ 0.1421 & 0.6686 & 0.1893 \\ 0.1008 & 0.4309 & 0.4683 \end{pmatrix} \end{array},$$

$$\vec{f}^{(CO)} = (0.2385, 0.5736, 0.1879).$$

Tables 10 and 11 characterize the four feasible scenarios for each of the industries. Industry s, starting from $\vec{f}^{(s)}$ is governed up to four subsequent quarters by $Q^{(1,s)}$ or $Q^{(8,s)}$, in the case of Table 10, while in Table 11 the percentages correspond to $Q^{(2,s)}$ or $Q^{(5,s)}$. The estimates contained in Tables 10 and 11 are comparable to those in Tables 8 and 9. Percentages in all tables containing macroeconomic scenarios reveal the same pattern: every scenario whose binary code contains more 1-s than 0-s implies a steady growth tendency, while those dominated by 0-s exhibit a steady decline. For the polar scenarios, the strongest steady growth and steady decline take place.

Table 10: Production. Balances in polar regimes.

		Number of subsequent quarters			
	Industry s	1	2	3	4
Polar upturn	IG	12.23	15.56	16.75	17.22
	CG	10.99	12.65	13.31	13.57
	CD	9.89	13.94	16.47	17.06
	CP	13.32	15.43	16.59	16.80
	CO	7.80	8.85	9.39	9.49
Polar downturn	IG	-7.87	-12.26	-14.04	-14.76
	CG	-1.99	-5.02	-6.11	-6.50
	CD	-5.72	-7.93	-9.05	-9.23
	CP	-4.79	-9.43	-11.75	-12.10
	CO	-0.55	-2.58	-3.60	-3.76

Table 11: Production. Balances in mixed regimes.

			Number of subsequent quarters			
Scenario	Probability	Industry s	1	2	3	4
$(1,1,0)$	0.1301	IG	7.10	8.78	9.50	9.79
		CG	8.49	9.29	9.67	9.83
		CD	6.60	9.55	11.65	12.25
		CP	10.43	11.51	12.18	12.34
		CO	7.57	8.53	9.04	9.12
$(0,1,1)$	0.1718	IG	8.00	9.59	10.06	10.19
		CG	5.75	5.86	5.97	6.03
		CD	3.65	4.71	5.13	5.16
		CP	7.69	8.04	8.31	8.35
		CO	7.32	8.19	8.64	8.71

The overall impression is that the estimates by industry are quite similar. The estimates for construction differ most from those for the manufacturing industries. This might be explained by the role of weather and the strength in the transmission of global business cycle fluctuations to the domestic economy. The effect of weather induces a seasonal pattern in the construction output. The effect of such fluctuations is quantitatively important for small open economies, but some sectors are more exposed to global business cycles than others. Construction is less exposed to the fluctuations in foreign demand than a typical manufacturing industry.

Let us briefly discuss a possible extension of the above model. We could make the model with sector differentiation richer by using industry-specific transition matrices $P^{(s)}$. If distinct transition matrices were estimated for different industries, then we could assume industry-specific macroeconomic factors for the equality constraints similar to (4). In this case we would assign a binary vector with three coordinates to each industry. With five industries, a macroeconomic scenario would thus be described by a binary vector with $3 \cdot 5 = 15$ coordinates. The number of potential scenarios would be equal to $2^{15} = 32768$. We could simplify the above model by assuming a single common macroeconomic factor for all industries. This would be appropriate if the blocks of three coordinates corresponding to different industries were identical, implying identical transition matrices for all industries. Given the complexity of the resulting optimization problem, we will not consider this extension in this paper, turning to a dynamic variant of the model with sector differentiation instead.

6 A dynamic micro-founded model with sector differentiation

Assume that the macroeconomic scenarios evolve as a time-homogeneous Markov chain whose transition matrix is $\mathcal{P}$. With $2^3 = 8$ possible macroeconomic scenarios, this is a 8×8 matrix. The corresponding likelihood function reads:

$$L(\vec{\rho}, q, d, \mathcal{P}) = \prod_{t=1}^{T} \sum_{k=1}^{8} [\vec{\rho}\mathcal{P}^{(t-1)}]_k \prod_{s=1}^{5} \prod_{i=1}^{3} \prod_{j=1}^{3} [q_{i,s} P_{i,j} + (1 - q_{i,s}) P_{i,j}(\chi_i^{(k)})]^{n_{i,j}^{(s)}(t)}.$$

For a given macroeconomic scenario, the random variables ζ_n are assumed to be independent across firms. The product in s assumes that, conditionally on a macroeconomic outcome, the respondents belonging to different industries respond independently. The vector $\vec{\rho}\mathcal{P}^{(t-1)}$ is the distribution of $\vec{\rho}$ after t time instants. $[\vec{\rho}\mathcal{P}^{(t-1)}]_k$ denotes the k-th coordinate of $\vec{\rho}\mathcal{P}^{(t-1)}$. Matrices q and $\mathcal{P}$, vector $\vec{\rho}$ and constant d have to be estimated.

Maximizing $\ln L(\vec{\rho}, q, d, \mathcal{P})$, there are linear inequality constraints: $\mathcal{P}_{i,j},\ q_{i,s},\ \rho_j,\ d \in [0, 1]$, linear equality constraints:

$$\sum_{i=1}^{8} \rho_i = 1, \quad \sum_{j=1}^{8} \mathcal{P}_{i,j} = 1, \quad i = 1, 2, \ldots, 8,$$

and non-linear (with respect to $\mathcal{P}$) equality constraints:

$$\begin{gathered}
[\vec{\rho}\mathcal{P}^{(t-1)}]_1 + [\vec{\rho}\mathcal{P}^{(t-1)}]_2 + [\vec{\rho}\mathcal{P}^{(t-1)}]_3 + [\vec{\rho}\mathcal{P}^{(t-1)}]_4 = P_{1,1}, \quad t = 1, 2, \ldots, T, \\
[\vec{\rho}\mathcal{P}^{(t-1)}]_1 + [\vec{\rho}\mathcal{P}^{(t-1)}]_2 + [\vec{\rho}\mathcal{P}^{(t-1)}]_5 + [\vec{\rho}\mathcal{P}^{(t-1)}]_6 - dP_{2,2} = P_{2,1}, \quad t = 1, 2, \ldots, T, \\
[\vec{\rho}\mathcal{P}^{(t-1)}]_1 + [\vec{\rho}\mathcal{P}^{(t-1)}]_3 + [\vec{\rho}\mathcal{P}^{(t-1)}]_5 + [\vec{\rho}\mathcal{P}^{(t-1)}]_7 = P_{3,1} + P_{3,2}, \quad t = 1, 2, \ldots, T.
\end{gathered} \tag{5}$$

The linear constraints state that coordinates of $\vec{\rho}$ form a probability distribution and that $\mathcal{P}$ is a Markovian transition matrix. We denote entries of $\mathcal{P}$ by $\mathcal{P}_{i,j}$. Relations (5) are dynamic counterparts of equalities (4): they are needed in order to guarantee that unconditionally the opinion-formation process is governed by P. As a consequence, the non-linear equality constraints implicitly imply that the macroeconomic dynamic has to be in equilibrium: the initial distribution $\vec{\rho}$ has to be the steady-state distribution of $\mathcal{P}$.

There is a simpler variant of the dynamic setting. Even if it does typically lead to an optimal solution, this heuristic seems to be conceptually appealing. A similar approach has been used for estimating the macroeconomic dynamic governing credit-rating migrations in Boreiko et al. (2017). Note that the just explained dynamic analyzes all possible macroeconomic scenarios. At the same time, the solution of the static model implies only four feasible scenarios. Confining to the feasible scenarios, we may consider a Markov chain with four states only. Then, the transition matrix $\mathcal{P}$ governing the macroeconomic dynamic has the dimension 4×4, reducing the number of unknowns from 64 to 16. The four scenarios are listed in the descending order of their likelihood: $(1,1,1)$, $(0,0,0)$, $(0,1,1)$, $(1,1,0)$, their probabilities being ρ_1, ρ_2, ρ_3, ρ_4, respectively. The corresponding reduction of the dynamic model formulated in this section is given in the Appendix. The following parameters were estimated:

$$q = \begin{array}{c c} & \begin{array}{ccccc} IG & CG & CD & CP & CO \end{array} \\ \begin{array}{c} op \\ nu \\ pe \end{array} & \begin{pmatrix} 0.7943 & 0.9211 & 0.8971 & 0.8817 & 0.9201 \\ 0.7077 & 0.8893 & 0.8533 & 0.7534 & 0.6726 \\ 0.8233 & 0.9260 & 0.9131 & 0.8179 & 0.9823 \end{pmatrix} \end{array}, \; d = 0.6424,$$

$$\vec{\rho} = (0.4144, 0.0813, 0.1230, 0.3813),$$

$$\mathcal{P} = \begin{array}{c c} & \begin{array}{cccc} (1,1,1) & (0,0,0) & (0,1,1) & (1,1,0) \end{array} \\ \begin{array}{c} (1,1,1) \\ (0,0,0) \\ (0,1,1) \\ (1,1,0) \end{array} & \begin{pmatrix} 0.9352 & 0.0000 & 0.0279 & 0.0369 \\ 0.0413 & 0.9449 & 0.0137 & 0.0001 \\ 0.0133 & 0.1338 & 0.8529 & 0.0000 \\ 0.1167 & 0.0558 & 0.0160 & 0.8115 \end{pmatrix} \end{array}, \; C = \begin{pmatrix} 1.0000 & 0.7784 & 0.5938 \\ 0.7784 & 1.0000 & 0.8461 \\ 0.5938 & 0.8461 & 1.0000 \end{pmatrix}.$$

The conditional probabilities $P_{2,2}(\chi_2)$ vary less than those estimated for the static model against their unconditional counterpart: $P_{2,2}(1)$ is 3.89 percent larger, while $P_{2,2}(0)$ is 6.21 percent smaller. The matrix q is quite similar to its static counterpart. The probabilities of the polar cases, 0.4144 and 0.3813, are markedly higher than their static counterparts. Together, these outcomes cover 79.57 percent of all quarters. In sum, the dynamic 'universe' is less dispersed than the static one. This conclusion is supported by larger off-diagonal elements of the correlation matrix C. The transition matrix governing the macroeconomic dynamic implies that no polar upturn can be followed by a polar downturn. In fact, $\mathcal{P}_{1,2} = 0$. A polar downturn is followed by a polar upturn with probability 0.0413. The state $(0,1,1)$ is more likely to deteriorate into a polar downturn than recover to a polar upturn. Indeed, 0.1338 is more than ten times larger than 0.0133. The steady-state distribution $(0.4146, 0.3810, 0.1230, 0.0814)$ of $\mathcal{P}$ almost coincides with the initial distribution $(0.4144, 0.3813, 0.1230, 0.0813)$ estimated for the dynamic model. The model suggests that the macroeconomic dynamic is close to its long-run equilibrium, which is given by the steady-state of the corresponding hidden Markov chain.

7 Summary and conclusions

The models presented in this paper offer tools for exploring common trends in categorical survey data. We have discussed several aggregate and micro-founded models. Some of these models are static while others admit dynamics in the underlying common trends. Using the total probability formula, dynamic models allow us to characterize multi-period scenarios. The aggregate models offer a birds eye perspective on the common conditions that can be assumed to influence every respondent in the sample. In our sample these common factors are assumed to be the macroeconomic conditions influencing the assessment of the current business situation and past production levels by Austrian firms. We characterize most of the quantitative results (observed or estimated) using a balance that expresses the surplus of optimists over pessimists. It appears that the estimates based on production show more internal consistency than those regarding the business situation. This probably owes to the fact that the question concerning production leaves less room for interpretation and less uncertainty.

The aggregate analysis allows us to identify two macroeconomic regimes that correspond to an economic upturn and a downturn, and introduce a dynamic in the succession of macroeconomic regimes using a hidden Markov chain. The estimated probability of occurrence of upturns is close to the frequency of quarters in which the observed production growth of the manufacturing sector has increased relative to the same quarter of the previous year. This observation confirms the assumed interpretation of the unobserved common factors and validates the model. Introducing regime switching at the aggregate level allows justifying the estimated probabilities of the two regimes and the steady-state distribution of the hidden Markov chain.

The more detailed micro-founded approach explicitly models the response of a firm. It is assumed that the response represents a noisy signal about the unobserved macroeconomic conditions common to all firms that responded in the same way. The fact that a survey question can be answered in three ways optimistic, neutral and pessimistic defines three opinion cohorts. This leads us to model three macroeconomic factors, each specific to one opinion cohort. Together, the three factors constitute a macroeconomic scenario. There are eight potential scenarios. We paid particular attention to the two polar scenarios, in which all types of firms (optimists, neutrals and pessimists) have experienced the same macroeconomic conditions, which can be favorable or adverse. These polar cases persist in all estimates presented in this paper. In the remaining six mixed scenarios some types of respondents experience favorable macroeconomic conditions while other types do not. In some models, some of the mixed scenarios are not observed because the estimated probability of their occurrence equals zero. In particular, the estimates of the static micro-founded model without sector differentiation show that two of the six mixed scenarios are not feasible. The number of feasible mixed scenarios can be further reduced to only two by allowing sector differentiation for a total of four feasible scenarios, two polar and two mixed.

The estimates of the micro-founded models offer a more fine-grained picture that requires some aggregation to make them comparable to that of the aggregate models. This leads us to the introduction of a representative agent (firm), whose probabilistic characteristics are obtained by aggregating individual responses across the entire pool of respondents. In general, the estimates of the micro-founded models are consistent with those of the aggregate models, and the estimates of the models with sector differentiation are similar to their counterparts of the models without the sector dimension. The balances under different macroeconomic scenarios vary by sector, with construction being distinct from manufacturing. The latter finding is unsurprising given the higher importance of seasonal fluctuation and the lesser importance of the fluctuations in foreign demand on construction output.

The most complex and computationally demanding type of the models considered here is the dynamic micro-founded model with sector differentiation. To reduce the number of parameters that need to be estimated, we have restricted the set of potential macroeconomic scenarios to the feasible scenarios identified in the static model with sector differentiation. The dynamic micro-founded model returns higher probabilities of the polar macroeconomic scenarios. The estimates of this model suggest near equilibrium dynamics of the underlying macroeconomic factors, a steady-state of the corresponding hidden Markov chain.

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Appendix: Likelihood functions and constraints

All estimators for the above models are obtained by maximizing the logarithm of a likelihood function subject to constraints. Since static models are particular cases of dynamic models, we will focus on the latter while indicating how a static model is obtained from a dynamic one.

We begin with the aggregate model based on a regime-switching matrix $\mathcal{P}$. Let m be the number of possible responses to a survey question (we have $m = 3$). The likelihood function is

$$L(P^U, P^D, \mathcal{P}, p) = \prod_{t=1}^{T}\{[(p,\ 1-p)\mathcal{P}^{t-1}]_1 \prod_{i=1}^{m}\prod_{j=1}^{m}(P^U_{i,j})^{n^t(i,j)} + [(p, 1-p)\mathcal{P}^{t-1}]_2 \prod_{i=1}^{m}\prod_{j=1}^{m}(P^D_{i,j})^{n^t(i,j)}\}.$$

The term $[(p,\ 1-p)\mathcal{P}^{t-1}]_k$ is the k-th coordinate of the vector $(p,\ 1-p)\mathcal{P}^{t-1}$, where

$$\mathcal{P} = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

The linear equality constraints are

$$\sum_{j=1}^{m} P^U_{i,j} = 1, \quad \sum_{j=1}^{m} P^D_{i,j} = 1, \quad i = 1, 2, \ldots, m,$$

whereas the linear inequality constrains are given by

$$P^U_{i,j} \geq P^D_{i,j} \ \text{ if } j < i \ \text{ and } P^D_{i,j} \geq P^U_{i,j} \ \text{ if } j > i, \quad i = 1, 2, \ldots, m;$$

$$\sum_{j=1}^{i-1}(P^U_{i,j} - P^D_{i,j}) + \sum_{j=i+1}^{m}(P^D_{i,j} - P^U_{i,j}) \geq \epsilon_i, \quad i = 1, 2, \ldots, m.$$

The thresholds ϵ_i are given non-negative numbers. If ϵ_i vanishes for some i, then the corresponding inequality in the second group follows from the inequalities involving i from the first group. The values $P^U_{i,j}$, $P^D_{i,j}$, p, α and β must belong to $[0, 1]$. The dynamic setting reduces to the static one when $\mathcal{P}$ becomes a 2×2 identity matrix I_2.

Turning to the micro-founded model, let us first generalize the formulas for the conditional probabilities to any number of states m. Let P be $m \times m$ transition matrix with entries $P_{i,j}$. Assuming that $1 \succ 2 \succ \ldots \succ m$, set

$$P_{i,j}(1) = \begin{cases} \frac{1}{P_i} P_{i,j} & \text{if } j < i, \\ \frac{\Delta_i}{P_i} P_{i,i} & \text{if } j = i, \\ 0 & \text{if } j > i; \end{cases} \quad \text{and } P_{i,j}(0) = \begin{cases} \frac{1}{1-P_i} P_{i,j} & \text{if } j > i, \\ \frac{1-\Delta_i}{1-P_i} P_{i,i} & \text{if } j = i, \\ 0 & \text{if } j < i. \end{cases}$$

Here, $P_i = P_{i,1} + P_{i,2} + \ldots + P_{i,i-1} + \Delta_i P_{i,i}$, $0 \leq \Delta_i \leq 1$, $i = 1, 2, \ldots, m$. Depending on whether Δ_i is larger or smaller than $\Delta_i^* = \frac{P_{i,1}+P_{i,2}+\ldots+P_{i,i-1}}{1-P_{i,i}}$, $P_{i,i}(1)$ $(P_{i,i}(0))$ will be larger (smaller) or smaller (larger) than $P_{i,i}$, $2 \leq i \leq m-1$. The probabilities of moving to the better (worse) states, $j < i$ $(j > i)$ increase relative to $P_{i,j}$ under favorable (adverse) macroeconomic conditions given by $\chi_i = 1$ $(\chi_i = 0)$ for the firms belonging to class i. Each transition probability is multiplied by a factor that exceeds one. The adjustment of these probabilities according to macroeconomic conditions depends on the parameters Δ_i, where $P_{i,i}$ is the probability that a firm will not change its opinion in the next quarter. There is no margin for the adjustment for $P_{1,1}$ and $P_{m,m}$. $\Delta_1 = 1$ and $\Delta_m = 0$ because $P_{1,i}(1)$ must be 0 for all $i > 1$ and $P_{m,j}(0)$ must be 0 for all $j < m$. The remaining Δ_i are estimated, together with the remaining model parameters, as a vector $\vec{d}$ with $m-2$ coordinates, such that $\Delta_i = d_{i-1}$. The percentage of variation of $P_{j,j}(\chi_j)$, $2 \leq j \leq m-1$, can be expressed as $\left(\frac{d_{j+1}}{P_j} - 1\right) \cdot 100$ if $\chi_j = 1$ and $\left(\frac{1-d_{j+1}}{1-P_j} - 1\right) \cdot 100$ if $\chi_j = 0$.

The likelihood function of the micro-founded setting is given by:

$$L(\vec{\rho}, q, \vec{d}, \mathcal{P}) = \prod_{t=1}^{T} \sum_{l=1}^{n_{\mathbf{BV}}} [\vec{\rho}^{(l)} \mathcal{P}^{t-1}]_i \prod_{s=1}^{S} \prod_{i=1}^{m} \prod_{j=1}^{m} [q_{i,s} P_{i,j} + (1 - q_{i,s}) P_{i,j}(\chi_i^{(l)})]^{n_{i,j}^{(s)}(t)}.$$

In the above formula, $n_{\mathbf{BV}}$ denotes the number of elements in a set $\mathbf{BV}$ of binary vectors with m coordinates, and S stands for the number of industry sectors considered. We set $S = 1$ in the case of no sector differentiation, so that q becomes a vector. For the static model, $\mathbf{BV}$ coincides with the set $\{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$ of all binary vectors with m coordinates. If $\mathbf{BV} = \{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$, then the numbering convention of Section 4 applies, i.e. the vector $(1, 1, \ldots, 1)$ is numbered by 1, while $(0, 0, \ldots, 0)$ receives the number 2^m. Otherwise, if $\mathbf{BV} \subset \{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$, then the binary vectors $\vec{\chi} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$ must be numbered according to $\vec{\chi}^{(l)}$, $l = 1, 2, \ldots, n_{\mathbf{BV}}$. For example, the estimates reported in Section 6 are obtained assuming that $\mathbf{BV}$ contains four vectors that are numbered in descending order of the probabilities assigned to the them by the solution of the respective static model. The l-th coordinate ρ_l of the $n_{\mathbf{BV}}$-vector $\vec{\rho}$ equals the probability assigned to the binary vector $\vec{\chi}^{(l)}$. The entry $\mathcal{P}_{l,k}$ of the $n_{\mathbf{BV}} \times n_{\mathbf{BV}}$ matrix $\mathcal{P}$ is the probability that the macroeconomic scenario encoded by $\vec{\chi}^{(l)}$ will be followed by the macroeconomic scenario corresponding to $\vec{\chi}^{(k)}$.

Linear inequality constraints read $\mathcal{P}_{l,k}$, $q_{i,s}$, ρ_l, $d_j \in [0, 1]$. Linear equality constraints are

$$\sum_{l=1}^{n_{\mathbf{BV}}} \rho_l = 1, \quad \sum_{k=1}^{n_{\mathbf{BV}}} \mathcal{P}_{l,k} = 1, \quad i = 1, 2, \ldots, n_{\mathbf{BV}}.$$

The nonlinear (with respect to $\mathcal{P}$) equality constraints are given by

$$\sum_{l=1}^{n_{\mathbf{BV}}} [\vec{\rho}^{(l)} \mathcal{P}^{(t-1)}]_i \chi_i^{(l)} = P_i, \quad i = 1, 2, \ldots, m, \quad t = 2, 3, \ldots, T.$$

Recall that P_i contains the parameter d_{i-1}, $i = 2, 3, \ldots, m-1$. The dynamic setting reduces to the static one when $\mathbf{BV} = \{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$ and $\mathcal{P}$ equals a $2^m \times 2^m$ identity matrix I_{2^m}.