Inequality in population weights, majority threshold and the inversion probability in the case of three states

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Abstract

Two-tier voting systems are prone to majority inversions, a situation in which the outcome of an election is not backed by a majority of popular vote. We study the inversion probability in a model with two candidates, three states and uniformly distributed fractions of supporters for each candidate. We show that the inversion probability in a two-tier voting system with three states eventually decreases with a majority threshold in the states and increases with the inequality in the size distribution of the states.

Key Words: inversion probability, majority threshold, two-tier voting

1 Introduction

Two-candidate elections conducted using two-tier electoral procedures are prone to majority inversions. Inversions occur when the electoral majority in the top tier contradicts the popular vote in the bottom tier. The result is that the outcome of an election does not represent the will of a majority of voters. Such a situation is illustrated for three equally-sized states in Figure 1 under simple majority rule. In the example, 'blue' wins the election by an electoral majority that is not backed by a simple majority of the popular vote. A majority inversion occurs, undermining the democratic legitimacy of the outcome.

Two-tier voting systems exemplified by the U.S. Electoral College and the British-type single-member-district majority systems have been the subject of formal and empirical study. Looking at the U.S. presidential election for a two-way election between the Democratic and the Republican candidates, several majority inversions have occurred in the past. For a recent empirical estimate of the probability of majority inversion in a U.S. presidential election, see Geruso, Spears and Talesara (2019).[1] The earliest formal analysis of inversion probability in a two-tier model with an odd number of equally-sized states was conducted by May (1948).[2] His model

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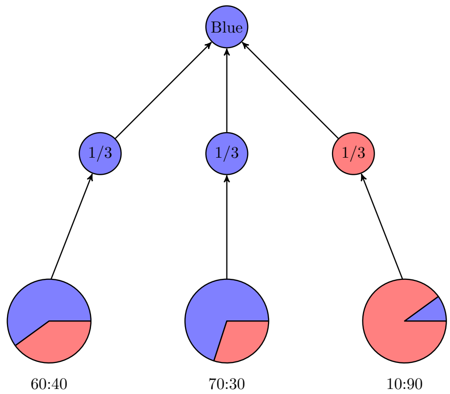
[1]Miller (2012) documents inversions in legislative elections in the Westminster parliamentary systems.

[2]For an alternative stochastic model, see Zaigraev and Kaniovski (2010) and the follow-up by Di Cecco (2011).

Figure 1: Example of majority inversion with three states.



Under simple majority rule, blue wins the election by a 2/3 majority of the electoral votes (top tier), despite having only a 140/300 minority of the popular vote (bottom tier).

assumes a discrete uniform distribution for the number of supporters of a certain candidate in each state, which became a continuous uniform distribution in the limiting case of infinitely many voters. The final feature of the model by May is the traditional majority threshold of 1/2, which is usually referred to as a *simple majority*. May has shown inversion probability in the case of three states to be equal to 1/8 under simple majority with equally-sized states. We relax this assumption of simple majority by a general majority threshold $\alpha \in (0, 1)$ and relax the assumption of equally-sized states. A majority threshold in excess of 1/2 should be interpreted as a *qualified majority* required to change the status quo. For example, passing a motion may require 2/3 of all votes, rather than the usual 1/2. Failing to achieve the required number of votes, the motion is rejected and the status quo remains.

For analytical tractability, we confine to the case of three states. The behavior of two-tier voting systems with three states under the May stochastic model of voting have been previously studied in Kaniovski and Zaigraev (2018), who show that the inversion probability increases with the size-discrepancy between the states. The inversion probability is a Schur-convex function of the vector of states' population weights. For an exposition of the theory of majorization and Schur-convexity, see Marshall, Olkin and Arnold (2011), whose formal concepts provide a foundation of the theory of inequality measurement by Yitzhaki and Schechtman (2013).

Briefly anticipating the main result, we find a trade-off between the majority threshold α and the inequality in the size distribution of states. The inversion probability in a two-tier voting system with three states decreases for sufficiently high α and increases with the inequality of the size distribution. This result is based on a complete analytical analysis of the inversion probability as a function of the population weights of the three states and α.

2 The result

In the baseline model, the voters face two alternatives, A and B. Let the number of states be n, where n is odd, and $n \geq 3$. The variations of the standard two-tier voting models studied in this paper rest on four assumptions. For each state $i = 1, \ldots, n$:

A.1 $w_i > 0$ the population weights, such that $\sum_{i=1}^n w_i = 1$;

A.2 $v_i > 0$ the voting weights, such that $\sum_{i=1}^n v_i = 1$ and $\max\{v_i\} < \frac{1}{2}$, whereby no state carries sufficient weight to dictate the outcome of the election;

A.3 $a_i \sim U(0, 1)$ the share of voters in state i who support A, where the uniformly distributed random variables $a_1, \ldots, a_n$ are independent;

A.4 $\alpha \in (0, 1)$ the majority threshold, i.e. the share of votes required to pass a motion in the bottom tier; while in the top tier the simple majority is needed.

The inversion probability is the sum of two probabilities

$$P\left(\sum_{i=1}^n w_i a_i < \alpha\ ,\ \sum_{j=1}^n v_j \mathbf{1}_{\{a_j > \alpha\}} > \frac{1}{2}\right) + P\left(\sum_{i=1}^n w_i a_i > \alpha\ ,\ \sum_{j=1}^n v_j \mathbf{1}_{\{a_j > \alpha\}} < \frac{1}{2}\right). \tag{1}$$

The above formula illustrates the two mutually exclusive prerequisites for an inversion – either A loses the bottom tier and wins the top one, or the other way around. In terms of an absolute number of votes, if the total population equals the total turnout m, then the total number of votes cast in state i equals mw_i, of which $mw_i a_i$ favor A. The total number of votes for A thus becomes $m \sum_{i=1}^n w_i a_i$, and A loses the bottom tier, if $m \sum_{i=1}^n w_i a_i < m\alpha$. Yet, A wins the top tier, if $\sum_{j=1}^n v_j \mathbf{1}_{\{a_j > \alpha\}} > 1/2$. This situation is illustrated for $n = 3$, $\alpha = 1/2$, $w_i = v_i = 1/3$ in Figure 1 (A is 'blue'). The value $\alpha = 1/2$ represents a simple majority.

Let $x_i = a_i - \alpha \sim U(-\alpha, 1 - \alpha)$ for $i = 1, \ldots, n$. Then, the inversion probability becomes:

$$\begin{aligned} P(\alpha, w_1, \ldots, w_n, v_1, \ldots, v_n) = P&\left(\sum_{i=1}^n w_i x_i < 0, \sum_{j=1}^n v_j \mathbf{1}_{\{x_j > 0\}} > \frac{1}{2}\right) \\ + P&\left(\sum_{i=1}^n w_i x_i > 0, \sum_{j=1}^n v_j \mathbf{1}_{\{x_j > 0\}} < \frac{1}{2}\right). \end{aligned} \tag{2}$$

Evidently, if $x_i \sim U(-\alpha, 1 - \alpha)$, then $-x_i \sim U(-(1 - \alpha), \alpha)$ for $i = 1, \ldots, n$. Multiplying the first inequalities under both probabilities by -1 and substituting x_i by $-x_i$, one can see that

$$P(\alpha, w_1, \ldots, w_n, v_1, \ldots, v_n) = P(1 - \alpha, w_1, \ldots, w_n, v_1, \ldots, v_n). \tag{3}$$

It is therefore sufficient to consider the case $\alpha \geq \frac{1}{2}$ only. In what follows, we assume $n = 3$. For $n = 3$, the assumption $\max\{v_i\} < \frac{1}{2}$ made in [A.2] implies that the inversion probability does not depend on the vector of voting weights (v_1, v_2, v_3). Under this assumption, the outcomes in exactly two of the three states must contradict the overall outcome for an inversion to occur, and this holds regardless of (v_1, v_2, v_3). Without any loss of generality, the population weights are ordered in a descending order, i.e. $w_1 \geq w_2 \geq w_3$.

Theorem 1. *The inversion probability assumes the following expressions:*
1) $w_3 \geq 1 - \alpha$:

$$P(\cdot) = \frac{3}{2}(1 - \alpha)^2(1 + \alpha) - \frac{(1 - \alpha)^3}{2}\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \frac{(1 - \alpha)^3(w_1^3 + w_2^3 + w_3^3)}{6w_1 w_2 w_3};$$

2) $w_1 \geq w_2 \geq 1-\alpha > w_3,\ w_3\alpha < w_2(1-\alpha)$:

$$P(\cdot) = 1-\alpha + \frac{(1-\alpha)(\alpha w_2 - 1 + 2\alpha - 2\alpha^2)}{2w_1} + \frac{(1-\alpha)(\alpha w_1 - 1 + 2\alpha - 2\alpha^2)}{2w_2} + \frac{[(1-\alpha)^3 + 3\alpha^3]w_3^2}{6w_1w_2};$$

3) $w_1 \geq w_2 \geq 1-\alpha > w_3,\ w_3\alpha \geq w_1(1-\alpha)$:

$$P(\cdot) = (1-\alpha)(1-2\alpha^2) + \frac{(1-\alpha)(2\alpha - 1 - \alpha w_2)}{2w_1} + \frac{(1-\alpha)(2\alpha - 1 - \alpha w_1)}{2w_2} + \frac{2(1-\alpha)^3 w_1^2}{6w_2w_3} + \frac{2(1-\alpha)^3 w_2^2}{6w_1w_3} + \frac{[(1-\alpha)^3 - \alpha^3]w_3^2}{6w_1w_2};$$

4) $w_1 \geq w_2 \geq 1-\alpha > w_3,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha)$:

$$P(\cdot) = (1-\alpha)^2(1+\alpha) + \frac{(1-\alpha)(2\alpha - 1 - \alpha w_2)}{2w_1} + \frac{(1-\alpha)(2\alpha - 1 - 2\alpha^2 + \alpha w_1)}{2w_2} + \frac{(1-\alpha)^3 w_2^2}{3w_1w_3} + \frac{(1-3\alpha+3\alpha^2)w_3^2}{6w_1w_2};$$

5) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_3\alpha \geq w_1(1-\alpha)$:

$$P(\cdot) = \frac{1}{2}(1-\alpha)(1+\alpha-3\alpha^2) + \frac{(1-\alpha)^3 w_1^2}{2w_2w_3} + \frac{[2(1-\alpha)^3 - \alpha^3]w_2^2}{6w_1w_3} + \frac{[2(1-\alpha)^3 - \alpha^3]w_3^2}{6w_1w_2} + \frac{\alpha(1-\alpha)(w_3\alpha - w_1(1-\alpha))}{2w_2} + \frac{\alpha(1-\alpha)(w_2\alpha - w_1(1-\alpha))}{2w_3} + \frac{(1-\alpha)(\alpha^2+\alpha-1)}{2w_1};$$

6) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_3\alpha < w_2(1-\alpha),\ w_2\alpha > w_1(1-\alpha)$:

$$P(\cdot) = \frac{1}{2}(1-\alpha)(1+\alpha-\alpha^2) + \frac{(1-\alpha)[\alpha(1+2w_2) - 1 - \alpha^2]}{2w_1} + \frac{(1-\alpha)^3 w_1^2}{6w_2w_3} - \frac{\alpha^3 w_2^2}{6w_1w_3} + \frac{\alpha(1-\alpha)(w_2\alpha - w_1(1-\alpha))}{2w_3} + \frac{\alpha(1-\alpha)(w_1(1-\alpha) - w_3\alpha)}{2w_2} + \frac{[2(1-\alpha)^3 + 3\alpha^3]w_3^2}{6w_1w_2};$$

7) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_3\alpha < w_2(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha),\ w_1 \leq \alpha$:

$$P(\cdot) = \frac{1}{2}(1-\alpha)(1+\alpha-\alpha^2) + \frac{(1-\alpha)[\alpha(1+2w_2) - 1 - \alpha^2]}{2w_1} + \frac{[2(1-\alpha)^3 + 3\alpha^3]w_3^2}{6w_1w_2} + \frac{\alpha(1-\alpha)((1-\alpha)w_1 - \alpha w_3)}{2w_2} + \frac{\alpha(1-\alpha)((1-\alpha)w_1 - \alpha w_2)}{2w_3} - \frac{(1-\alpha)^3 w_1^2}{6w_2w_3} + \frac{\alpha^3 w_2^2}{6w_1w_3};$$

8) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_3\alpha < w_2(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha),\ w_1 > \alpha$:

$$P(\cdot) = \frac{1}{2} + \frac{2\alpha(1-\alpha)(1+w_2) - 1}{2w_1} + \frac{(1-3\alpha+3\alpha^2)w_3^2}{3w_1w_2};$$

9) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha),\ w_1 \leq \alpha$:

$$P(\cdot) = \frac{1}{2}(1-\alpha)(1+\alpha-3\alpha^2) - \frac{(1-\alpha)(1-\alpha-\alpha^2)}{2w_1} + \frac{\alpha(1-\alpha)((1-\alpha)w_1 - \alpha w_3)}{2w_2} + \frac{\alpha(1-\alpha)((1-\alpha)w_1 - \alpha w_2)}{2w_3} - \frac{(1-\alpha)^3 w_1^2}{6w_2w_3} + \frac{[2(1-\alpha)^3 + \alpha^3]w_2^2}{6w_1w_3} + \frac{[2(1-\alpha)^3 + \alpha^3]w_3^2}{6w_1w_2};$$

10) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha),\ w_1 > \alpha$:

$$P(\cdot) = \frac{1}{2}(1-2\alpha^2+2\alpha^3) - \frac{1-2\alpha+2\alpha^3}{2w_1} + \frac{(1-\alpha)^3 w_2^2}{3w_1w_3} + \frac{(1-\alpha)^3 w_3^2}{3w_1w_2};$$

11) $w_1 \geq 1-\alpha > w_2 \geq w_3,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha) < w_2\alpha$:

$$P(\cdot) = \frac{1}{2}(1-\alpha)(1+\alpha-3\alpha^2) - \frac{(1-\alpha)(1-\alpha-\alpha^2)}{2w_1} + \frac{[2(1-\alpha)^3+\alpha^3])w_3^2}{6w_1w_2} + \frac{(1-\alpha)^3 w_1^2}{6w_2w_3}$$
$$+ \frac{\alpha(1-\alpha)(w_2\alpha - w_1(1-\alpha))}{2w_3} + \frac{[2(1-\alpha)^3-\alpha^3]w_2^2}{6w_1w_3} + \frac{\alpha(1-\alpha)(w_1(1-\alpha)-w_3\alpha)}{2w_2};$$

12) $w_1 < 1-\alpha,\ w_3\alpha \geq w_1(1-\alpha)$:

$$P(\cdot) = -\frac{3}{2}\alpha(1-\alpha)(2\alpha-1) + \frac{\alpha(1-\alpha)(2\alpha-1)}{2}\left(\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}\right)$$
$$+ \frac{[3(1-\alpha)^3-\alpha^3](w_1^3+w_2^3+w_3^3)}{6w_1w_2w_3};$$

13) $w_1 < 1-\alpha,\ w_3\alpha < w_2(1-\alpha),\ w_2\alpha > w_1(1-\alpha)$:

$$P(\cdot) = -\frac{\alpha(1-\alpha)(2\alpha-1)}{2} + \frac{\alpha(1-\alpha)(2\alpha-1)}{2w_3} + \frac{[(1-\alpha)^3-\alpha^3]w_1^2}{6w_2w_3} + \frac{[(1-\alpha)^3-\alpha^3]w_2^2}{6w_1w_3}$$
$$+ \frac{(1-3\alpha+3\alpha^2)w_3^2}{2w_1w_2} + \frac{\alpha(1-\alpha)(w_2-w_3)}{2w_1} + \frac{\alpha(1-\alpha)(w_1-w_3)}{2w_2};$$

14) $w_1 < 1-\alpha,\ w_3\alpha < w_2(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha)$:

$$P(\cdot) = \frac{\alpha(1-\alpha)}{2}\left(\frac{w_2-w_3}{w_1}+\frac{w_1-w_3}{w_2}+\frac{w_1-w_2}{w_3}\right) + \frac{(1-3\alpha+3\alpha^2)(-w_1^3+w_2^3+3w_3^3)}{6w_1w_2w_3};$$

15) $w_1 < 1-\alpha,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha),\ w_2\alpha \leq w_1(1-\alpha)$:

$$P(\cdot) = -\frac{1}{2}\alpha(1-\alpha)(2\alpha-1) + \frac{\alpha(1-\alpha)(2\alpha-1)}{2w_1} + \frac{\alpha(1-\alpha)(w_1-w_3)}{2w_2} + \frac{\alpha(1-\alpha)(w_1-w_2)}{2w_3}$$
$$- \frac{(1-3\alpha+3\alpha^2)w_1^2}{6w_2w_3} + \frac{[3(1-\alpha)^3+\alpha^3]w_2^2}{6w_1w_3} + \frac{[3(1-\alpha)^3+\alpha^3]w_3^2}{6w_1w_2};$$

16) $w_1 < 1-\alpha,\ w_2(1-\alpha) \leq w_3\alpha < w_1(1-\alpha) < w_2\alpha$:

$$P(\cdot) = -\alpha(1-\alpha)(2\alpha-1) + \frac{\alpha(1-\alpha)(2\alpha-1)}{2w_1} + \frac{\alpha(1-\alpha)(2\alpha-1)}{2w_3} + \frac{\alpha(1-\alpha)(w_1-w_3)}{2w_2}$$
$$+ \frac{[(1-\alpha)^3-\alpha^3]w_1^2}{6w_2w_3} + \frac{[3(1-\alpha)^3-\alpha^3]w_2^2}{6w_1w_3} + \frac{[3(1-\alpha)^3+\alpha^3]w_3^2}{6w_1w_2}.$$

3 Summary

The implications of Theorem 1 are illustrated in Figure 2. The inequality of the population weights increases along the abscissa (left panel). This inequality can be summarized using a

Gini coefficient, a popular measure of inequality that assumes a particularly simple form in the case of three weights, such that $w_1 + w_2 + w_3 = 1$:

$$G(w_1, w_2, w_3) = \frac{2(w_1 - w_3)}{3},$$

taking the minimal value of 0 for $w_1 = w_2 = w_3 = 1/3$, and the maximal value of $2/3$ for $w_1 = 1$ and $w_2 = w_3 = 0$. The inversion probability tends to $\alpha(1-\alpha)$, as $G(\cdot)$ tends to $2/3$. The second parameter that changes in the scenarios from left to right is the majority threshold α, starting from $1/2$ and tending to 1 (right panel).

Turning to the inversion probability, we can formulate the following conclusions:

- Fixing the majority threshold $\alpha = 1/2$ and increasing the disparity of population weights increases the inversion probability (left panel, black line). This finding confirms the main result in Kaniovski and Zaigraev (2018).

- Turning to the right panel of Figure 2, we obtain the following corollary for the case of equal population weights (right panel, black line):

  Corollary 1.

  $$P\left(\alpha, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \begin{cases} \frac{3}{2}\alpha(1-\alpha)(2\alpha-1) - \frac{1}{2}(2\alpha-1)^3 + (1-\alpha)^3 & \text{if } \alpha \in [\frac{1}{2}, \frac{2}{3}), \\ 3(1-\alpha)^2(2\alpha-1) + \frac{1}{2}(1-\alpha)^3 & \text{if } \alpha \in [\frac{2}{3}, 1). \end{cases}$$

  When α increases from $\frac{1}{2}$ to 1, $P(\alpha, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ increases from $\frac{1}{8}$ to $\frac{17}{128}$ (for $\alpha = \frac{5}{8}$) and then decreases from $\frac{17}{128}$ to 0.

- It is easy to see that for $\alpha = 1/2$, the two summands in the formula for the inversion probability are equal. In general, these summands will not be equal. The first one will be larger than the second one for small Gini coefficients, whereas the converse may be true for large Gini coefficients.

- When the population weights are unequal but fixed (right panel), increasing the majority threshold α decreases the inversion probability. However, the inversion probability as a function of α is not strictly decreasing in general, possibly achieving an interior maximum for moderate weight discrepancies, i.e. when the Gini coefficient is small.

- In general, there exists a trade-off between the majority threshold and the inequality in the population weights. The inversion probability in a two-tier voting system with three states increases with the discrepancy in the size distribution of states and decreases for sufficiently high majority thresholds. This implies the existence of a combination of population weights and a majority threshold, such that the inversion probability equals that under the assumptions of the May model. May has shown the inversion probability to be equal to 1/8 under simple majority rule ($\alpha = 1/2$) with three equally-sized states ($w_1 = w_2 = w_3$), as is verified by Corollary 1.

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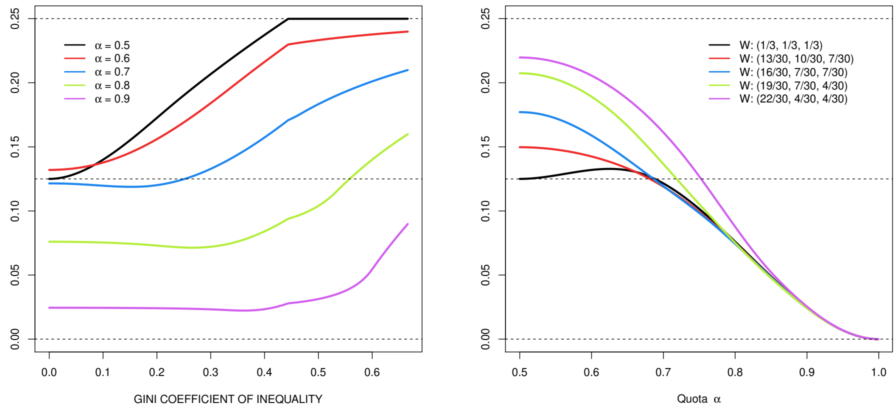
Figure 2: Inequality in population weights, majority threshold and the inversion probability.



The left panel shows how the inversion probability varies for five values of α when the inequality of population weights increases, such that the Gini coefficient rises from 0 to 2/3. The right panel shows the variation of the inversion probability with α for equal population weights (black) and several combinations of unequal population weights.

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