Exact bounds on the probability of at least k successes in n exchangeable Bernoulli trials as a function of correlation coefficients

Alexander Zaigraev* Serguei Kaniovski†

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Abstract

We compute the minimum and maximum of the probability of k or more successes in n exchangeable Bernoulli trials as a function of the correlation coefficients. This probability finds wide application in reliability and decision theory. Since the probability is linear in the coefficients, finding the minimum and maximum requires solving linear programming problems. We show that the maximum can be lower than certainty (no certain success), whereas the minimum can be higher than zero (positive residual risk).

Key Words: exchangeable Bernoulli trials, k-out-of-n system, linear programming

1 Introduction

We use the parametrizations of the joint probability distribution of exchangeable Bernoulli random variables by Bahadur (1961) and George and Bowman (1995) to compute the minimum and maximum of the probability of k or more successes in n exchangeable Bernoulli trials as a function of correlation coefficients. In doing so we characterize the distributions corresponding to the minimum and maximum, which may not be unique. Since the probability is linear in the coefficients, finding the minimum and maximum requires solving linear programming problems with constraints imposed by the non-negativity of the joint distribution.

The probability of k or more successes in n binary trials plays an important role in the reliability of k-out-of-n systems, and our presentation will focus on this application. A k-out-of-n system consists of n components, such that each component is either functioning, or it has failed. The reliability of a system is defined as the probability that the system will function. Factors that may lead to dependent component performance include the influence of a common

*Nicolaus Copernicus University
Faculty of Mathematics and Computer Science
Chopin str. 12/18, 87-100 Toruń, Poland
Tel: +48 56 6112944
Email: alzaig@mat.uni.torun.pl

†Austrian Institute of Economic Research (WIFO)
P.O. Box 91, A-1103 Vienna, Austria
Tel: +43 1 7982601 231
Email: serguei.kaniovski@wifo.ac.at

operating environment, and the fact that failure of one component may increase the strain on the remaining components, leading to the failure cascades (Lindley and Singpurwalla 2002). For a survey, see Hsieh (2003). The reliability of a consecutive k-out-of-n system with exchangeable components has been computed in Eryilmaz and Demir (2007).

The same probability finds an application in decision theory. The literature on Condorcet's Jury Theorem studies the expertise of a group of experts. The experts cast their vote in favor of one of two alternatives. Individual votes are aggregated into a collective judgment using a voting rule k, for example simple majority rule: $k = (n+1)/2$ for odd n. Stochastic independence cannot be reconciled with commonalities and differences in experts' preferences, information asymmetries and strategic behavior, as these factors will induce correlations between the votes. Recent literature studies the probability of the correct decision in the case of correlated votes. For a survey, see Kaniovski and Zaigraev (2009).

Exchangeability imposes a dependence structure that is a) analytically tractable, and b) can be maintained on a priori grounds in the absence of specific knowledge about the numerous mixed moments of the joint distribution.

2 Notation and preliminaries

Two commonly studied k-out-of-n systems are the G system and the F system. A k-out-of-n:G system functions if k or more of its components function, whereas a k-out-of-n:F system fails if at least k components fail. A k-out-of-n:F is therefore equivalent to an $n-k+1$-out-of-n:G system.

Let component i's state be a realization x_i of a Bernoulli random variable X_i, such that

$$x_i = \begin{cases} 1 & \text{if component } i \text{ functions;} \\ 0 & \text{if component } i \text{ fails.} \end{cases}$$

The reliability of i's component is measured by its probability of being functional $p_i = P(X_i = 1)$.

A vector of states $\mathbf{x} = (x_1, \dots, x_n)$ is called a system profile. There will be 2^n such profiles. The structure function of a k-out-of-n:G system is given by

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^n x_i \geq k; \\ 0 & \text{if} \quad \sum_{i=1}^n x_i < k. \end{cases}$$

Let $\pi_{\mathbf{x}}$ be the probability of occurrence of $\mathbf{x}$, and $t(\mathbf{x}) = n - \sum_{i=1}^n x_i$ be the number of failing components. The reliability of such a system can be defined as

$$R_{n,p}^k = E[\phi(\mathbf{X})] = P(\phi(\mathbf{X}) = 1) = \sum_{i=0}^{n-k} \pi_{\mathbf{x}:t(\mathbf{x})=i}, \quad \text{where} \quad 1 \leq k \leq n, \tag{1}$$

and E denotes the expected value.

Bahadur (1961) proved that $\pi_{\mathbf{x}}$ is uniquely determined by n marginal probabilities and $2^n - n - 1$ correlation coefficients, defined as:

$$\begin{array}{ll} \text{order } 2 & c_{i,j} = E[Z_i Z_j] \quad \text{for all} \quad 1 \leq i < j \leq n; \\ \text{order } 3 & c_{i,j,r} = E[Z_i Z_j Z_r] \quad \text{for all} \quad 1 \leq i < j < r \leq n; \\ & \dots \\ \text{order } n & c_{1,2,\dots,n} = E[Z_1 Z_2 \dots Z_n], \end{array}$$

where $Z_i = (X_i - p_i)/\sqrt{p_i(1-p_i)}$ for $i = 1, 2, \ldots, n$.

A sequence of random variables is exchangeable if the joint probability distribution is invariant under the permutation of its arguments. The sequence of binary random variables $X_1, X_2, \ldots, X_n$ is exchangeable if

$$P(X_1 = x_1, \ldots, X_r = x_r) = P(X_{\sigma(1)} = x_1, \ldots, X_{\sigma(r)} = x_r)$$

for any $1 \leq r \leq n$, and any permutation $\sigma = (\sigma(1), \ldots, \sigma(r))$ of the indices $\{1, 2, \ldots, r\}$. In this case $\pi_{\mathbf{x}}$ depends on the total number of failing components, not on their order. Since their number ranges from 0 to n, there can be at most $n+1$ distinct probabilities, so that (1) becomes

$$R^k_{n,p} = \sum_{i=0}^{n-k} C^i_n \pi_i. \tag{2}$$

Exchangeability requires the equality of the marginal probabilities and the equality of all correlation coefficients of the same order. We thus define the correlation coefficient of the i-th order as

$$c_i = \frac{E[(X_1 - p) \ldots (X_i - p)]}{\sqrt{p^i(1-p)^i}} \quad \text{for} \quad i = 2, 3, \ldots, n.$$

The assumption of exchangeability constrains the correlation coefficients to ensure non-negativity of the probability. These constraints are not known in general. Bounds on c_2 when $c_3 = c_4 = \cdots = c_n = 0$ are given in Bahadur (1961) and Kaniovski and Zaigraev (2009), and illustrated in Stefanescu and Turnbull (2003). Bahadur's parametrization for exchangeable random variables is given in Zaigraev and Kaniovski (2009).

In this paper, we use a more compact parametrization by George and Bowman (1995). Let $\lambda_i = P(X_1 = 1, X_2 = 1, \ldots, X_i = 1)$, $i = 1, 2, \ldots, n$ (of course, $\lambda_1 = p$), and $\lambda_0 = 1$. Then,

$$\pi_i = \sum_{j=0}^{i} (-1)^j C^j_i \lambda_{n-i+j} = \Delta^i(\lambda_{n-i}), \tag{3}$$

where C^i_n denotes the binomial coefficient, and $\Delta^i(\lambda_{n-i})$ denotes the i-th finite difference of λ_{n-i}. The correlation coefficients can then be recovered as

$$c_i = \frac{\sum_{j=0}^{i-2} (-1)^j C^j_i p^j \lambda_{i-j} + (-1)^{i-1}(i-1)p^i}{\sqrt{p^i(1-p)^i}} \quad \text{for} \quad i = 2, 3, \ldots, n. \tag{4}$$

The equivalence of the two parametrizations allows us to switch between a representation of the joint distribution of n exchangeable binary variables with the marginal p in terms of $c_2, \ldots, c_n$ and $\lambda_2, \ldots, \lambda_n$. We shall use this fact in proving our theorem.

Since the correlation coefficients enter $R^k_{n,p}$ linearly, and the non-negativity of the probabilities π_i imposes $n+1$ linear constraints, finding correlation coefficients that minimize or maximize $R^k_{n,p}$ for given n and p requires to solving the following linear programming problems:

$$R^k_{n,p}(c_2, \ldots, c_n) = R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) \to \min \quad \text{subject to} \quad \pi_i \geq 0, \quad i = 0, 1, \ldots, n; \tag{5}$$

$$R^k_{n,p}(c_2, \ldots, c_n) = R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) \to \max \quad \text{subject to} \quad \pi_i \geq 0, \quad i = 0, 1, \ldots, n. \tag{6}$$

The above problems can be written in the standard form as:

$$\mathbf{a}^T\boldsymbol{\lambda} \to \min \quad \text{subject to} \quad \mathbf{A}\boldsymbol{\lambda} \le \mathbf{b}; \tag{7}$$

$$\mathbf{a}^T\boldsymbol{\lambda} \to \max \quad \text{subject to} \quad \mathbf{A}\boldsymbol{\lambda} \le \mathbf{b}, \tag{8}$$

where $\boldsymbol{\lambda} = (\lambda_2, \dots, \lambda_n)$, $\mathbf{a} \in \mathbb{R}^{n-1}$, $\mathbf{b} \in \mathbb{R}^{n+1}$ and $\mathbf{A} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$; and in the dual form as:

$$\mathbf{b}^T\boldsymbol{\theta} \to \min \quad \text{subject to} \quad \mathbf{A}^T\boldsymbol{\theta} = -\mathbf{a}, \quad \theta_i \ge 0, \quad i = 0, 1, \dots, n; \tag{9}$$

$$\mathbf{b}^T\boldsymbol{\theta} \to \min \quad \text{subject to} \quad \mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}, \quad \theta_i \ge 0, \quad i = 0, 1, \dots, n. \tag{10}$$

The Duality Theorem says that if $\boldsymbol{\lambda}^*$ and $\boldsymbol{\theta}^*$ are the solutions of the primal Problem (7) and the corresponding dual Problem (9), then $\mathbf{a}^T\boldsymbol{\lambda}^* = -\mathbf{b}^T\boldsymbol{\theta}^*$; while if $\boldsymbol{\lambda}^*$ and $\boldsymbol{\theta}^*$ are the solutions of the primal Problem (8) and the corresponding dual Problem (10), then $\mathbf{a}^T\boldsymbol{\lambda}^* = \mathbf{b}^T\boldsymbol{\theta}^*$.

The above primal linear programming problems involve $n-1$ variables and $n+1$ constraints. Typically, problems of this size can only be solved numerically. A closer look at the dual problems reveals, however, that the solutions of the primal problems satisfy the systems of linear equations formed by vanishing probabilities.

3 The theorem

Theorem. *Let $R_{n,p}^k$ be the probability of at least k successes in n exchangeable Bernoulli trials having marginal probability p. Then,*

$$\max\left\{\frac{np-k+1}{n-k+1}, 0\right\} \le R_{n,p}^k \le \min\left\{\frac{np}{k}, 1\right\}.$$

Proof: See Appendix.

The Theorem finishes the bounds on the reliability of a k-out-of-n:G system. Analogous bounds for a k-out-of-n:F system are obtained by substituting $n-k+1$ for k. The validity of the theorem can be illustrated on the special cases of a series ($k = n$) and a parallel system ($k = 1$). The reliability of a series G-system with exchangeable components cannot exceed that of a single component p, whereas the reliability of a parallel G-system cannot be lower than p. The opposite holds for series and parallel F-systems. Indeed, substituting $k = n$ or $k = 1$ gives

$$\max\{1 - n(1-p), 0\} \le R_{n,p}^n \le p; \tag{n-out-of-n:G}$$

$$p \le R_{n,p}^1 \le \min\{np, 1\}; \tag{1-out-of-n:G}$$

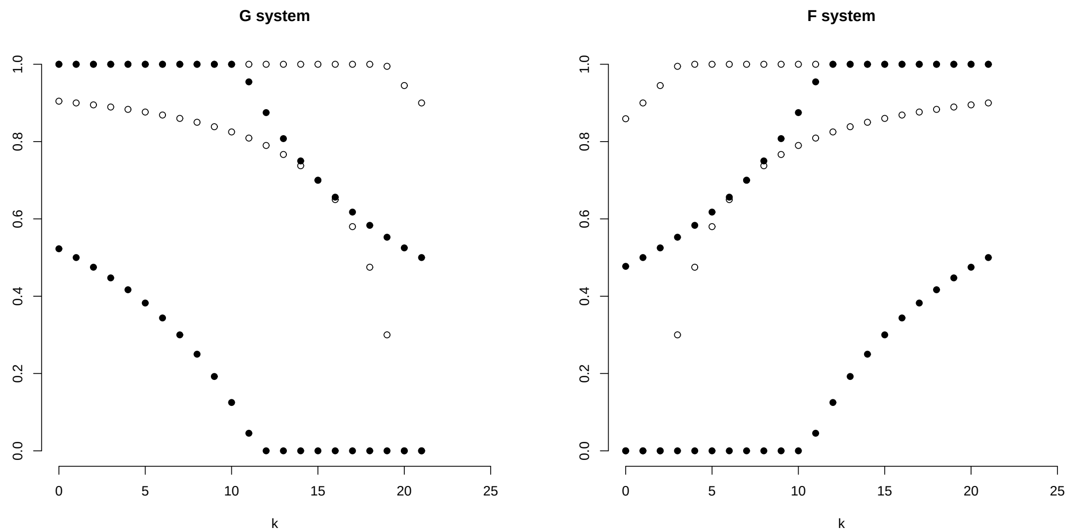
$$p \le R_{n,p}^n \le \min\{np, 1\}. \tag{n-out-of-n:F}$$

$$\max\{1 - n(1-p), 0\} \le R_{n,p}^1 \le p. \tag{1-out-of-n:F}$$

Figure 1 illustrates how $\min R_{n,p}^k$ and $\max R_{n,p}^k$ vary in k for a given n and p.

In the proof of the Theorem, we express the solutions of the linear programming problems in terms of probabilities $\{\pi_i\}$. Alternatively, we can use (3) and (4) to express the solutions in terms of the probabilities $\lambda_2, \lambda_3, \dots, \lambda_n$, or the correlation coefficients $c_2, c_3, \dots, c_n$.

Figure 1: $\min R^k_{n,p}$ and $\max R^k_{n,p}$ as functions of k



For $n = 21$, $p = 0.5$ (bullet) and $p = 0.9$ (circle).

A Proofs

Proof. Minimum. In Problems (7) and (8), the objective function $R^k_{n,p}$ and the vector $\mathbf{a}$ will be different depending on whether $k = 1$, or $k \geq 2$, whereas in dual Problems (9) and (10), $\mathbf{b} = (0, \ldots, 0, p, 1 - np)^T$, and

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & \ldots & -1 & C^1_{n-1} & -C^2_n \\ 0 & 0 & 0 & \ldots & C^1_{n-2} & -C^2_{n-1} & C^3_n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & -1 & 2 & \ldots & (-1)^n C^{n-3}_{n-2} & (-1)^{n-1} C^{n-2}_{n-1} & (-1)^n C^{n-1}_n \\ -1 & 1 & -1 & \ldots & (-1)^{n-1} & (-1)^n & (-1)^{n-1} \end{pmatrix}.$$

In each case, we use (3) to express $R^k_{n,p}$ in terms of $\boldsymbol{\lambda}$.

Case 1: $k = 1$. We have,

$$R^1_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = 1 - \pi_n = np + \sum_{j=2}^{n} (-1)^{j+1} C^j_n \lambda_j = np + \sum_{j=2}^{n} a_j \lambda_j,$$

where $a_j = (-1)^{j+1} C^j_n,\ j = 2, 3, \ldots, n$. The solution to the system of equations $\mathbf{A}^T \boldsymbol{\theta} = -\mathbf{a}$ is

$$\theta_i = C^i_{n-1}\theta_{n-1} - (n-1-i) C^i_n \theta_n - (n-1-i) C^i_n, \quad \text{for} \quad i = 0, \ldots, n-2,$$

whereby the set of constraints $\mathbf{A}^T \boldsymbol{\theta} = -\mathbf{a}$, $\theta_i \geq 0$, $i = 0, 1, \ldots, n$, becomes

$$C^i_{n-1}\theta_{n-1} - (n-1-i) C^i_n \theta_n - (n-1-i) C^i_n \geq 0, \quad \text{for} \quad i = 0, \ldots, n-2, \quad \theta_{n-1} \geq 0, \quad \theta_n \geq 0.$$

For the above constraints to hold, it suffices that $\theta_0 \geq 0$ and $\theta_n \geq 0$. Consequently, the solutions of dual Problem (9) coincide with the solutions of the following simpler problem:

$$\mathbf{b}^T\boldsymbol{\theta} = p\theta_{n-1} + (1-np)\theta_n \to \min \text{ subject to } \theta_{n-1} - (n-1)\theta_n - (n-1) \geq 0, \quad \theta_n \geq 0,$$

whose solution $(\theta^*_{n-1}, \theta^*_n)$ lies on the vertex defined by $\theta_{n-1} - (n-1)\theta_n - (n-1) = 0$, and $\theta_n = 0$. Since $p\theta^*_{n-1} + (1-np)\theta^*_n = (n-1)p$ at $(\theta^*_{n-1}, \theta^*_n) = (n-1, 0)$, we have

$$\min_{\lambda_2,\lambda_3,\ldots,\lambda_n} R^1_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = np - (n-1)p = p.$$

The solution is given by $\pi_0 = p$, $\pi_1 = 0, \ldots, \pi_{n-1} = 0$, $\pi_n = 1-p$.

Case 2: $k \geq 2$. In this case, we have

$$R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = \sum_{i=0}^{n-k} C^i_n \sum_{j=0}^{i} (-1)^j C^j_i \lambda_{n-i+j} = \sum_{i=2}^{n} a_i \lambda_i,$$

where $a_2 = \cdots = a_{k-1} = 0$, $a_{k+i} = (-1)^i C^{n-k-i}_n C^i_{k-1+i}$ for $i = 0, 1, \ldots, n-k$.

The solution to the system of equations $\mathbf{A}^T\boldsymbol{\theta} = -\mathbf{a}$ is

$$\begin{aligned}
&\theta_i = C^i_{n-1}\theta_{n-1} - (n-1-i)C^i_n\theta_n + C^i_n, \quad \text{for} \quad i = 0, \ldots, n-k;\\
&\theta_i = C^i_{n-1}\theta_{n-1} - (n-1-i)C^i_n\theta_n, \quad \text{for} \quad i = n-k+1, \ldots, n-2.
\end{aligned}$$

Therefore, the set of constraints $\mathbf{A}^T\boldsymbol{\theta} = -\mathbf{a}$, $\theta_i \geq 0$ for $i = 0, 1, \ldots, n$, becomes

$$\begin{aligned}
&C^i_{n-1}\theta_{n-1} - (n-1-i)C^i_n\theta_n + C^i_n \geq 0, \quad \text{for} \quad i = 0, \ldots, n-k;\\
&C^i_{n-1}\theta_{n-1} - (n-1-i)C^i_n\theta_n \geq 0, \quad \text{for} \quad i = n-k+1, \ldots, n-2;\\
&\theta_{n-1} \geq 0 \quad \text{and} \quad \theta_n \geq 0.
\end{aligned}$$

For them to hold, it suffices that $\theta_0 \geq 0$, $\theta_{n-k+1} \geq 0$, and $\theta_n \geq 0$. Consequently, the solutions of dual Problem (9) coincide with the solutions of the problem:

$$\begin{aligned}
&\mathbf{b}^T\boldsymbol{\theta} = p\theta_{n-1} + (1-np)\theta_n \to \min \quad \text{subject to}\\
&\quad \theta_{n-1} - (n-1)\theta_n + 1 \geq 0;\\
&\quad C^{n-k+1}_{n-1}\theta_{n-1} - (k-2)C^{n-k+1}_n\theta_n \geq 0;\\
&\quad \theta_n \geq 0.
\end{aligned}$$

Case 2a: $p < \frac{k-1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta^*_{n-1}, \theta^*_n) = (0, 0)$ and $p\theta^*_{n-1} + (1-np)\theta^*_n = 0$. Therefore, $\min_{\lambda_2,\lambda_3,\ldots,\lambda_n} R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = 0$, and $\pi_0 = 0, \ldots, \pi_{n-k} = 0$, $\pi_{n-k+1} \geq 0, \ldots, \pi_n \geq 0$.

Case 2b: $p = \frac{k-1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at any $(\frac{n(k-2)}{k-1}\theta_n, \theta_n)$, where $\theta_n \in [0, \frac{k-1}{n-k+1}]$, and at those points $p\theta_{n-1} + (1-np)\theta_n = 0$. Thus, $\min_{\lambda_2,\lambda_3,\ldots,\lambda_n} R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = 0$. If $\theta_n \in (0, \frac{k-1}{n-k+1}]$, then $\pi_{n-k+1} = \frac{1}{C^{n-k+1}_n}$, whereas the remaining π_i's vanish. If $\theta_n = 0$, then $\pi_0 = 0, \ldots, \pi_{n-k} = 0, \pi_{n-k+1} \geq 0, \ldots, \pi_n \geq 0$.

Case 2c: $p > \frac{k-1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta^*_{n-1}, \theta^*_n) = (\frac{n(k-2)}{n-k+1}, \frac{k-1}{n-k+1})$ and $p\theta^*_{n-1} + (1-np)\theta^*_n = \frac{k-1-np}{n-k+1}$. Therefore, $\min_{\lambda_2,\lambda_3,\ldots,\lambda_n} R^k_{n,p}(\lambda_2, \lambda_3, \ldots, \lambda_n) = \frac{np-k+1}{n-k+1}$. The solution is given by $\pi_0 = \frac{np-k+1}{n-k+1}, \pi_1 = 0, \ldots, \pi_{n-k} = 0, \pi_{n-k+1} = \frac{1-p}{C^{n-k}_{n-1}}, \pi_{n-k+2} = 0, \ldots, \pi_n = 0$.

Combining all these cases completes the proof for the minimum. $\square$

Proof. Maximum. Problem (8) is solved in a similar manner. Switch to the solution of correspondent dual Problem (10) and consider the system of equations $\mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}$.

Case 1: $k = 1$. Here, as in the first part, $a_j = (-1)^{j+1}C_n^j,\ j = 2, 3, \ldots, n$. The solution to the system of equations $\mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}$ is

$$\theta_i = C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n + (n-1-i)C_n^i, \quad \text{for} \quad i = 0, \ldots, n-2,$$

whereby the set of constraints $\mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}$, $\theta_i \geq 0$, $i = 0, 1, \ldots, n$, becomes

$$C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n + (n-1-i)C_n^i \geq 0, \quad \text{for} \quad i = 0, \ldots, n-2, \quad \theta_{n-1} \geq 0, \quad \theta_n \geq 0.$$

For the above constraints to hold, it suffices that $\theta_0 \geq 0,\ \theta_{n-1} \geq 0$ and $\theta_n \geq 0$. Consequently, the solutions of dual Problem (10) coincide with the solutions of the following simpler problem:

$$\mathbf{b}^T\boldsymbol{\theta} = p\theta_{n-1} + (1-np)\theta_n \to \min \text{ subject to } \theta_{n-1} - (n-1)\theta_n + (n-1) \geq 0,\ \theta_{n-1} \geq 0,\ \theta_n \geq 0.$$

Case 1a: $p < \frac{1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta_{n-1}^*, \theta_n^*) = (0, 0)$ and $p\theta_{n-1}^* + (1-np)\theta_n^* = 0$. Therefore, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^1(\lambda_2, \lambda_3, \ldots, \lambda_n) = np$. The joint distribution is given by: $\pi_0 = 0, \ldots, \pi_{n-2} = 0, \pi_{n-1} = p, \pi_n = 1 - np$.

Case 1b: $p = \frac{1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at any $(0, \theta_n)$, where $\theta_n \in [0, 1]$, and at those points $p\theta_{n-1} + (1-np)\theta_n = 0$. Therefore, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^1(\lambda_2, \lambda_3, \ldots, \lambda_n) = 1$. If $\theta_n \in [0, 1)$, then $\pi_{n-1} = \frac{1}{n}$, whereas the remaining π_i's vanish. If $\theta_n = 1$, then $\pi_0 \geq 0, \ldots, \pi_{n-1} \geq 0, \pi_n = 0$.

Case 1c: $p > \frac{1}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta_{n-1}^*, \theta_n^*) = (0, 1)$ and $p\theta_{n-1}^* + (1-np)\theta_n^* = 1 - np$. Therefore, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^1(\lambda_2, \lambda_3, \ldots, \lambda_n) = 1$. The joint distribution is given by: $\pi_0 \geq 0, \ldots, \pi_{n-1} \geq 0, \pi_n = 0$.

Case 2: $k \geq 2$. In this case, as in the first part, $a_2 = \cdots = a_{k-1} = 0$, $a_{k+i} = (-1)^i C_n^{n-k-i} C_{k-1+i}^i$ for $i = 0, 1, \ldots, n-k$. The solution to the system of equations $\mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}$ is

$$\begin{aligned}
\theta_i &= C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n - C_n^i, \quad \text{for} \quad i = 0, \ldots, n-k; \\
\theta_i &= C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n, \quad \text{for} \quad i = n-k+1, \ldots, n-2.
\end{aligned}$$

Therefore, the set of constraints $\mathbf{A}^T\boldsymbol{\theta} = \mathbf{a}$, $\theta_i \geq 0$ for $i = 0, 1, \ldots, n$, becomes

$$\begin{aligned}
&C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n - C_n^i \geq 0, \quad \text{for} \quad i = 0, \ldots, n-k; \\
&C_{n-1}^i\theta_{n-1} - (n-1-i)C_n^i\theta_n \geq 0, \quad \text{for} \quad i = n-k+1, \ldots, n-2; \\
&\theta_{n-1} \geq 0 \quad \text{and} \quad \theta_n \geq 0.
\end{aligned}$$

For them to hold, it suffices that $\theta_0 \geq 0$, $\theta_{n-k} \geq 0$, and $\theta_n \geq 0$. Consequently, the solutions of dual Problem (10) coincide with the solutions of the problem:

$$\begin{aligned}
&\mathbf{b}^T\boldsymbol{\theta} = p\theta_{n-1} + (1-np)\theta_n \to \min \quad \text{subject to} \\
&\quad \theta_{n-1} - (n-1)\theta_n - 1 \geq 0; \\
&\quad C_{n-1}^{n-k}\theta_{n-1} - (k-1)C_n^{n-k}\theta_n - C_n^{n-k} \geq 0; \\
&\quad \theta_n \geq 0.
\end{aligned}$$

Case 2a: $p < \frac{k}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta_{n-1}^*, \theta_n^*) = (\frac{n}{k}, 0)$ and $p\theta_{n-1}^* + (1-np)\theta_n^* = \frac{np}{k}$. Therefore, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^k(\lambda_2, \lambda_3, \ldots, \lambda_n) = \frac{np}{k}$. In terms of the joint probability distribution the solution can be written as: $\pi_0 = 0, \ldots, \pi_{n-k-1} = 0$, $\pi_{n-k} = \frac{p}{C_{n-1}^{n-k}}, \pi_{n-k+1} = 0, \ldots, \pi_{n-1} = 0, \pi_n = 1 - \frac{np}{k}$.

Case 2b: $p = \frac{k}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at any $(\frac{n(k-1)}{k}\theta_n + \frac{n}{k}, \theta_n)$, where $\theta_n \in [0,1]$, and at those points $p\theta_{n-1}+(1-np)\theta_n = 1$. Thus, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^k(\lambda_2, \lambda_3, \ldots, \lambda_n) = 1$. If $\theta_n \in [0,1)$, then $\pi_{n-k} = \frac{1}{C_n^{n-k}}$, whereas the remaining π_i's vanish. If $\theta_n = 1$, then $\pi_0 \geq 0, \ldots, \pi_{n-k} \geq 0, \pi_{n-k+1} = 0, \ldots, \pi_n = 0$.

Case 2c: $p > \frac{k}{n}$. The function $\mathbf{b}^T\boldsymbol{\theta}$ attains a minimum at $(\theta_{n-1}^*, \theta_n^*) = (n, 1)$ and $p\theta_{n-1}^*+(1-np)\theta_n^* = 1$. Therefore, $\max_{\lambda_2,\lambda_3,\ldots,\lambda_n} R_{n,p}^k(\lambda_2, \lambda_3, \ldots, \lambda_n) = 1$. In terms of the joint probability distribution the solution can be written as: $\pi_0 \geq 0, \ldots, \pi_{n-k} \geq 0, \pi_{n-k+1} = 0, \ldots, \pi_n = 0$.

Combining all these cases completes the proof for the maximum. □

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