Representation-Compatible Power Indices

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Abstract

This paper studies power indices based on average representations of a weighted game. If restricted to account for the lack of power of null voters, average representations become coherent measures of voting power, with power distributions being proportional to the distribution of weights in the average representation. This makes these indices representation-compatible, a property not fulfilled by classical power indices. In this paper we introduce two computationally cheaper alternatives to the existing representation-compatible power indices, and study the properties of a family that now comprises four measures.

Keywords: average representation; power index; proportionality between weights and power

1 Introduction

We commonly represent a weighted voting game using an integer-valued vector of voting weights $(w_1, w_2, \ldots, w_n)$ and an integer quota q. The vector of weights conveys the number of votes each of the n voters commands. The game involves each voter casting all her votes as a bloc either for or against a motion. The motion is passed if the total number of votes cast by the voters in favor of the motion is greater than or equal to q; otherwise, the motion is rejected. In this paper, we introduce two new measures of power for weighted games, and study the properties of a family of representation-compatible power indices that now comprises four measures.

Writing a weighted games as $[q; w_1, w_2, \ldots, w_n]$, a representation $(q; w_1, w_2, \ldots, w_n)$ conveys the set of winning coalitions of the weighted game. For example, winning coalitions for a game represented by $(51; 47, 46, 5, 2)$ are

$$\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}.$$

This set allows us to compute the voting power of the voters. A voter is critical to a winning coalition if the coalition becomes a losing one should she withdraw her support. No voter is critical in a coalition of all voters $\{1,2,3,4\}$. The first voter is critical in $\{1,2\}$, $\{1,2,4\}$, $\{1,3\}$ and $\{1,3,4\}$. The largest three voters acting together can pass a motion, but none of them would be critical to the success of the coalition $\{1,2,3\}$. Since each of the three large voters is critical in exactly four winning coalitions, they should be equally powerful. The smallest voter is a null

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voter, because she is irrelevant to the success of any coalition. The above considerations suggest $(1/3, 1/3, 1/3, 0)$ as a plausible distribution of voting power. Note that the three voters are equally powerful despite their unequal weights. The distribution of weights $(0.47, 0.46, 0.05, 0.02)$ is markedly different from the distribution of power. This difference would decrease if we chose to represent the above game using $(51; 34, 33, 33, 0)$. It would completely disappear if we chose the rational-valued representation $(2/3; 1/3, 1/3, 1/3, 0)$.

Although the set of winning coalitions uniquely defines a weighted game, there are infinitely many representations consistent with any given set of winning coalitions. If we adhere to a total of 100 votes among three voters and a quota of 51, then there will be 8924 integer-valued weight distributions consistent with the power vector $(1/3, 1/3, 1/3, 0)$. If the quota itself is considered to be part of the specification, then there will be 79800 possibilities to represent the game. Scaling the quota and weights in any of these representations by the same factor would preserve the set of winning coalitions and consequently define the same weighted game.

Since any admissible representation defines the game, the multiplicity of representations has no bearing on the definition of a weighted game. The multiplicity poses a problem if we want to design a weighted game (a voting institution such as a parliament) with a given distribution of power or to compare the distribution of power to the distribution of weights. Whereas imposing additional criteria can reduce the number of feasible representations and even identify a unique appropriate representation in sufficiently small, weighted games, the multiplicity makes unclear which representation the power distribution should be compared to.[1] The average representations can reasonably be compared to power distributions of various power indices, as they uniquely summarize the set of admissible representations.

The proportionality of power and weight has received significant attention in the literature. It has been noted that none of the classical power indices yield power distributions that coincide with weight distributions for all weighted games. The observation that the distribution of voting power is different from the distribution of voting weight has been one of the motivating factors behind the development of the theory of power measurement (Felsenthal and Machover 2005). Two recent theoretical studies provide conditions for which the weight distribution and the power distribution coincide. These results are available for the Banzhaf index by Houy and Zwicker (2014) and the nucleolus by Kurz, Napel and Nohn (2014). An exception is the Minimum Sum Representation Index (MSRI) introduced in Freixas and Kaniovski (2014), which is specifically designed to fulfill proportionality. Among the infinitely many representations of a weighted voting game, the minimal sum representation has the smallest sum of voting weights. The power value according to the MSRI equals the share in the sum of voting weights in a minimal sum representation.

The average representations come very close to being valid measures of power for weighted games. If restricted to account for the lack of power of null voters, average representations become coherent measures of voting power, with power distributions being proportional to the distribution of weights in the average representation. Restricting the polytope implied by the set of minimal winning and maximal losing coalitions yields average representations that are null-revealing. These restricted average representations satisfy Freixas and Gambarelli's (1997) coherency criteria for power indices, which are essentially equivalent to the widely accepted 'minimal adequacy postulate' by Felsenthal and Machover (1998) (p. 222). Kaniovski and Kurz

[1]We could minimize the total sum of integral voting weights to obtain a unique representation for $n \leq 7$, as in Freixas and Molinero (2009), or Freixas and Kurz (2014). For other options, see Application 9.9 in Crama and Hammer (2011).

(2015) introduced two representation-compatible power indices, called AWI and ARI, which are based on restricted average representations. The main drawback of these indices is the computational burden of numerical integration on highly-dimensional convex polytopes. In this paper we introduce two new representation-compatible power indices that will be computationally cheaper in many weighted voting games.

Representation-compatible indices possess another attractive property. By ascribing equal power to all members of an equivalence class of voters defined by the Isbell desirability relation, they rank voters according to their effect on the decisiveness of a coalition, where a more 'effective' voter represents a more desirable addition to a coalition. For example, a game represented by $(51; 47, 46, 5, 2)$ has two equivalence classes, which segregate the three larger voters from the null voter. Introducing restrictions based on the equivalence classes to the polytope implied by the set of minimal winning and maximal losing coalitions leads to another pair of power indices. These new power indices are typically computationally less expensive than the existing representation-compatible power indices, because the number of equivalence classes is typically smaller than the number of voters. Together with AWI and ARI, the new indices, called AWTI and ARTI, complete the family of representation-compatible power measures studied here.

In the next section, we recall the preliminaries required to define representation-compatible power indices. Section 3 defines the indices, verifies their coherency as measures of power and discusses computational issues. The computation of representation-compatible indices involves the integration of monomials on highly-dimensional polytopes with rational vertices (Section 4). Section 5 first compares the power distributions generated by representation-compatible indices to power distributions according to the Banzhaf (1965) and Shapley and Shubik (1954) indices in small weighted games, and then discusses their vulnerabilities to certain anomalies, commonly referred to as voting paradoxes. Section 6 discusses some aspects of the integer-valued representations that have been used by Freixas and Kaniovski (2014) to construct a power index. It turns out that average representations and the MSRI are related. The final section offers concluding remarks and ideas for future research.

2 Notation and preliminaries

2.1 Simple games and weighted games

A (monotonic) simple game is the most general type of binary voting game.

Definition 2.1 *A simple game v is a mapping $v : 2^n \to \{0, 1\}$, where $N = \{1, \ldots, n\}$ is the set of voters, such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \le v(T)$ for all $S \subseteq T \subseteq N$ (monotonicity).*

A subset $S \subseteq N$ is called a coalition of v. There are 2^n such coalitions in a simple game with n voters. A coalition S is winning if $v(S) = 1$, and losing if $v(S) = 0$. The monotonicity ensures that enlarging a winning coalition cannot make it a losing one, which is a sensible assumption.

A winning coalition S is called a *minimal winning coalition* if none of its proper subsets are winning. Similarly, a losing coalition T is called a *maximal losing coalition* if none of its proper supersets are losing. The set of minimal winning coalitions $\mathcal{W}^m$, or the set of maximal losing coalitions $\mathcal{L}^m$, uniquely defines a simple game. For the game represented by $(51; 47, 46, 5, 2)$, the set of minimal winning coalitions is given by $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. We define a simple game using the set of minimal winning coalitions as opposed to the set of winning coalitions, as the former definition is more compact. A weighted game is a simple game that admits a representation $(q; w_1, w_2, \ldots, w_n)$.

Definition 2.2 *A simple game v is weighted, if there exist real numbers $w_1, \ldots, w_n \geq 0$ and $q > 0$, such that*

$$\sum_{s \in S} w_s \geq q \quad \Longleftrightarrow \quad v(S) = 1,$$

for all $S \subseteq N$. We write: $(N, v) = [q; w_1, \ldots, w_n]$.

In this paper, we consider weighted games, as this type of binary voting games is most relevant to the applied power measurement and institutional design. A common institution that uses weighted voting for decision making is the shareholder assembly in a corporation. The voting weight of a shareholder equals the number of ordinary shares she holds. This example also includes voting by the member states in multilateral institutions such as the World Bank and the International Monetary Fund. In the political arena, voting in parliaments can be viewed as a weighted game, provided party discipline is absolute. The frequently studied voting in the Council of Ministers of the European Union can be viewed, with some simplification of the double-majority voting rule stipulated by the Lisbon Treaty, as a weighted game. In the examples above, the voting weights are non-negative integers. The conditions required for a simple game to be a weighted game have been studied extensively in the literature.[2]

2.2 Equivalence classes of voters

The equivalence classes serve two purposes. They partition the set of voters according to their effect on the decisiveness of coalitions. Any reasonable measure of voting power should, therefore, recognize the equivalence classes. Second, while each majority game has an infinite number of representations, the number of possible partitions of all games with a given number of voters is finite. The qualifier 'for all games' then stands for 'all feasible partitions of players in classes'. Our comparisons between power indices presented in Section 5 were obtained using this set of games, where each game is defined by its minimum sum representation.[3]

Definition 2.3 *Given a simple game v, we say that two voters $i, j \in N$ are equivalent, denoted by $i \sim j$, if we have $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.*

The relation $\sim$ is an equivalence relation and partitions the set of voters N into, say t, disjoint subsets $N_1, \ldots, N_t$ - the equivalence classes of voters. Roughly speaking, adding voter i instead of voter j to any coalition S will have the same or better effect on its decisiveness, making i a more desirable addition for the voters comprising S. This motivates the desire to design power indices that convey the membership in equivalence classes in a clear way. The number of equivalence classes can be significantly smaller than the number of voters. For example, the game $[51; 47, 46, 5, 2]$ has two equivalence classes, as we remarked in the introduction, whereas the game $[9; 5, 4, 3, 2, 1]$ has five equivalence classes, each voter in a class of its own.

The following three types of voters deserve special attention.

Definition 2.4 *Given a simple game v, a voter $i \in S$ with $v(S) = v(S \cup \{i\})$ for all $S \subseteq N \backslash \{i\}$ is called a null voter.*

A null voter is powerless, because she has no bearing on the success of a coalition.

[2] See, Taylor and Zwicker (1999). For a survey, see Chapter 9.8 in Crama and Hammer (2011).

[3] See, Freixas and Pons (2010).

Definition 2.5 *Given a simple game v, a voter $i \in N$ such that i is contained in all minimal winning coalitions is called a vetoer.*

Any voter in a minimal winning coalition is critical to the success of the coalition. This means that a voter present in all minimal winning coalitions has the power of a veto.

Definition 2.6 *Given a simple game v, a voter $i \in N$ such that $\{i\}$ is the unique minimal winning coalition is called a dictator.*

Being a dictator is the strongest form of having a veto. A dictator has all the power, rendering all other voters nulls.

Let us now recall some well-known facts about representations of weighted games:

Lemma 2.7 *Each weighted game v admits a representation $(q, w_1, \ldots, w_n)$ with $w_1, \ldots, w_n \geq 0$, $q > 0$, and*

(1) $\sum_{i=1}^n w_i = 1$, $q \in (0, 1]$;

(2) $\sum_{i=1}^n w_i = 1$, and $w_i = 0$ for all nulls $i \in N$;

(3) $q \in \mathbb{N}$, $w_i \in \mathbb{N}$;

(4) $q \in \mathbb{N}$, $w_i \in \mathbb{N}$, $w_i = w_j$ for all $i \sim j$, and $w_i = 0$ for all nulls $i \in N$.

We call (1) a normalized representation, and (3) an integer representation. Whenever we have $w_i = w_j$ for all $i \sim j$, we say that the representation is *type-revealing.* Note that $w_i \neq w_j$ implies $i \not\sim j$. A representation with $w_i = 0$ for all nulls $i \in N$ is called *null-revealing.* Again, $w_i \neq 0$ implies that player i is a non-null player. Given a general (integer) representation, the problem of verifying that a voter is a null is co-NP-complete (Theorem 4.4 in Chalkiadakis, Elkind and Wooldridge (2011)). If all weights of the given integer representation are comparatively small, then the generating function approach offers an efficient way of finding null voters in weighted voting games, see, for example, Bilbao, Fernández, Losada and López (2000).

2.3 Coherent power measures

Let $\mathcal{S}_n$ denote the set of simple games on n voters, and $\mathcal{W}_n \subset \mathcal{S}_n$ the set of weighted games on n voters.

Definition 2.8 *A power index for $\mathcal{C} \in \{\mathcal{S}_n, \mathcal{W}_n\}$ is a mapping $g : \mathcal{C} \to \mathbb{R}^n$, where n denotes the number of voters in each game of $\mathcal{C}$.*

We define a vector-valued power index by defining its element g_i, the voting power of voter i. A power index should satisfy the following essential properties:

Definition 2.9 *Let $g : \mathcal{C} \to \mathbb{R}^n = (g_i)_{i \in N}$ be a power index for $\mathcal{C}$. We say that*

(1) g is symmetric if for all $v \in \mathcal{C}$ and any bijection $\tau : N \to N$ we have $g_{\tau(i)}(\tau v) = g_i(v)$, where $\tau v(S) = v(\tau(S))$ for all $S \subseteq N$;

(2) g is positive if $g_i(v) \geq 0$ and $g(v) \neq 0$ for all $v \in \mathcal{C}$;

(3) g is efficient if $\sum_{i=1}^n g_i(v) = 1$ for all $v \in \mathcal{C}$;

(4) g satisfies the null property if for all $v \in \mathcal{C}$ and all nulls i of v we have $g_i(v) = 0$.

Any positive power index g can be made efficient by rescaling: $g'_i(v) = g_i(v)/\sum_{i=1}^{n} g_i(v)$. Rescaling turns the Penrose-Banzhaf absolute measure into the Banzhaf index. The Banzhaf index and the Shapley-Shubik index have all the above properties.

In addition to the above properties, any reasonable measure of voting power should recognize the equivalence classes of voters. To formalize this property, we need the notion of desirability introduced in Isbell (1956):

Definition 2.10 *Given a simple v, we write $i \succeq j$, if we have $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$ and say that voter i is at least as desirable as voter j.*

We can have $i \succeq j$ and $j \succeq i$, if and only if $i \sim j$. In this case, voters i and j are equivalent in the sense of belonging to the same equivalence class. We say $i \succ j$, if $i \succeq j$ and $i \not\sim j$. In an arbitrary simple game, we can have $i \not\succeq j$ and $j \not\succeq i$. In this case, the two voters $i, j \in N$ are incomparable. To exclude this possibility, a class of games narrower than simple games but still more general than weighted voting games has been proposed by Isbell (1956) and elaborated in Einy and Lehrer (1989), Carreras and Freixas (1996), and Taylor and Zwicker (1999).

Definition 2.11 *A simple game v is called complete if we have $i \succeq j$ or $j \succeq i$ (including both possibilities) for all voters $i, j \in N$.*

Taylor and Pacelli (2008) offer a test of completeness. A simple game is complete if it is swap robust, or if a one-for-one exchange of players between any two winning coalitions S and T leaves at least one of the two coalitions winning. One of the players in the swap must belong to S but not T, and the other must belong to T but not S.

It is important to emphasize that all weighted games are complete, so that the $\succeq$-relation induces a complete, or total, ordering of the voters. Given a representation $(q; w_1, \ldots, w_n)$, $w_i \geq w_j$ implies $i \succeq j$, and $w_i = w_j$ implies $i \sim j$. The implication $i \succ j$ from $w_i > w_j$ is only valid if the given representation preserves types formed by the partition of voters according to the equivalence relationship. However, $i \succ j$ implies $w_i > w_j$ in any representation.

Definition 2.12 *A power index $g : \mathcal{C} \to \mathbb{R}^n = (g_i)_{i \in N}$ for $\mathcal{C}$ satisfies strong monotonicity if we have $g_i(v) > g_j(v)$ for all $v \in \mathcal{C}$ and all voters with $i \succ j$ in v.*

According to Freixas and Gambarelli (1997), a power index is coherent if it satisfies the four properties of Definition 2.9 and is strongly monotonic. Strong monotonicity ensures that the power index recognizes the equivalence classes of voters and respects the desirability relation.

3 Representation-compatible power indices

The indices studied in this paper use the notions of *feasibility* and *representation-compatibility.* The first notion applies to a normalized vector of voting weights, whereas the second notion applies to a representation of a weighted voting game.

Definition 3.1 *Given a weighted game v, a vector $(q; w_1, \ldots, w_n)$ is a representation of v if $v = [q; w_1, \ldots, w_n]$. A weight vector $(w_1, \ldots, w_n)$ is called feasible for v if there exists a quota q such that $(q; w_1, \ldots, w_n)$ is a representation of v.*

For a normalized vector of weights to be feasible, it must fulfill the linear inequality constraints imposed by the set of minimal winning coalitions and the set of maximal losing coalitions.

Lemma 3.2 *The set of all normalized weight vectors* $w \in \mathbb{R}^n_{\geq 0}$, $\sum_{i=1}^n w_i = 1$ *being feasible for a given weighted game* v *is given by*

$$\sum_{i \in S} w_i > \sum_{i \in T} w_i$$

for all pairs (S, T)*, where* S *is a minimal winning and* T *is a maximal losing coalition of* v*.*

Similarly, for a representation to be valid, or compatible with a given weighted game, it must fulfill the linear inequality constraints imposed by the set of minimal winning coalitions and the maximal losing coalitions of the game.

Lemma 3.3 *The set of all normalized representations* $(q; w) \in \mathbb{R}^{n+1}_{\geq 0}$, $q \in (0, 1]$, $\sum_{i=1}^n w_i = 1$ *representing a given weighted game* v *is given by*

$$\sum_{i \in S} w_i \geq q, \quad \sum_{i \in T} w_i < q$$

for all minimal winning coalitions S *and all maximal losing coalitions* T*.*

We remark that there exist arbitrarily close rational-valued approximations to every real-valued feasible vector and every real-valued representation. We can obtain integer-valued vectors and representations by multiplying with the least common multiple of the denominators.

The two sets of linear inequalities of Lemma 3.2 and Lemma 3.3 define convex polytopes in Euclidean space if we replace the strict inequalities by non-strict inequalities. The following known lemma shows that this modification is justified and the dimensions of the corresponding polytopes are n in the case of the Lemma 3.3 and $n-1$ in the case of Lemma 3.2. For completeness, we give a short sketch of a proof.

Lemma 3.4 *For each weighted game* v *there exist positive real numbers* $\tilde{q}, \tilde{w}_1, \ldots, \tilde{w}_{n-1}$*, and a parameter* $\alpha > 0$*, such that* $\left(\tilde{q} + \delta_0, \tilde{w}_1 + \delta_1, \ldots, \tilde{w}_{n-1} + \delta_{n-1}, 1 - \sum_{i=1}^{n-1} (\tilde{w}_i + \delta_i)\right)$ *is a normalized representation of* v *for all* $\delta_i \in [-\alpha, \alpha]$, $0 \leq i \leq n-1$.

Proof. Let $(q, w_1, \ldots, w_n)$ be an integer representation of v. Consequently, the weight of each winning coalition is at least q, and the weight of each losing coalition is at most $q - 1$. Since $\big((n+1)q, (n+1)w_1 + 1, \ldots, (n+1)w_n\big)$ is also an integer representation of v, we additionally assume, without any loss of generality, that $w_i \geq 1$ for all $1 \leq i \leq n$. One can easily check that $\left(q - \frac{2}{5} + \tilde{\delta}_0, w_1 + \tilde{\delta}_1, \ldots, w_n + \tilde{\delta}_n\right)$ is a representation of v for all $\tilde{\delta}_i \in \left[-\frac{1}{5n}, \frac{1}{5n}\right]$, $0 \leq i \leq n$. With $s = \sum_{i=1}^n w_i$, let $\tilde{q} = \frac{1}{s} \cdot \left(q - \frac{2}{5}\right)$, $\tilde{w}_i = \frac{1}{s} \cdot w_i$ for all $1 \leq i \leq n-1$, and $\alpha = \frac{1}{10ns}$. $\square$

The four power indices for weighted games studied in this paper respects the proportionality of power and weight. We call such power indices *representation-compatible.*

Definition 3.5 *A power index* $g : \mathcal{W}_n \to \mathbb{R}^n$ *for weighted games on* n *voters is called representation-compatible if* $(g_1(v), \ldots, g_n(v))$ *is feasible for all* $v \in \mathcal{W}_n$*.*

Roughly speaking, a power index is representation compatible if for every weighted voting game v, the power vector can be used as weights in a representation $(q; w_1, \ldots, w_n)$. The existing power measures are not representation-compatible in general. For example, the Banzhaf index (BZI) and the Shapley-Shubik index (SSI) are representation-compatible for $n \leq 3$ only.

Table 1 compares all the weighted games with up to three voters in minimum sum integer representations to the respective power distribution according to the BZI and SSI. Pick the game $[3; 2, 1, 1]$. Its SSI power vector is given by $\left(\frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right)$. The power vector is feasible according to Definition 3.1, because we can find a quota, here $\frac{5}{6}$, such that the weighted voting games $[3; 2, 1, 1]$ and $\left[\frac{5}{6}; \frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right]$ have identical sets of minimal winning coalitions and maximal losing coalitions - they are two distinct representations of the same weighted voting game. If SSI would have this property for any weighted voting game, it would be representation-compatible. But this is not the case. For $n \geq 4$, one can easily find examples in which the SSI vector is not representation-compatible. For example, take the representation $(3; 2, 1, 1, 1)$. The corresponding SSI vector is given by $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$. Since $\{2, 3, 4\}$ is a winning coalition with weight $\frac{1}{2}$, and $\{1\}$ is a losing coalition with weight $\frac{1}{2}$, the SSI vector cannot be a representation of the game. The same counter-example also applies for the BZI, since in this game the two power vectors coincide. Houy and Zwicker (2014) characterize the set of representations that is compatible with the BZI in a general weighted game.

It is not a coincidence that some power vectors in Table 1 occur several times. This follows from duality.

Definition 3.6 *Let $v : 2^N \to \{0, 1\}$ be a simple game and $\mathcal{W}$ its set of winning coalitions, $\mathcal{L}$ its set of losing coalitions. By $v^d : 2^N \to \{0, 1\}$, with $v^d(S) = 1 - v(N \backslash S)$ for all $S \subseteq N$, we denote its dual game.*

Table 1: Representation-compatibility of the BZI and the SSI for $n \leq 3$.

Game	BZI	SSI	Game	BZI	SSI
$[1; 1]$	$[1; 1]$	$[1; 1]$	$[2, 1, 1, 0]$	$\left[\frac{6}{6}; \frac{3}{6}, \frac{3}{6}, \frac{0}{6}\right]$	$\left[\frac{6}{6}; \frac{3}{6}, \frac{3}{6}, \frac{0}{6}\right]$
$[1; 1, 0]$	$\left[\frac{2}{2}; \frac{2}{2}, \frac{0}{2}\right]$	$\left[\frac{2}{2}; \frac{2}{2}, \frac{0}{2}\right]$	$[1; 1, 1, 1]$	$\left[\frac{2}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$	$\left[\frac{2}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$
$[1; 1, 1]$	$\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right]$	$\left[\frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right]$	$[2; 1, 1, 1]$	$\left[\frac{4}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$	$\left[\frac{4}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$
$[2, 1, 1]$	$\left[\frac{2}{2}; \frac{1}{2}, \frac{1}{2}\right]$	$\left[\frac{2}{2}; \frac{1}{2}, \frac{1}{2}\right]$	$[3; 1, 1, 1]$	$\left[\frac{6}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$	$\left[\frac{6}{6}; \frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right]$
$[1; 1, 0, 0]$	$\left[\frac{6}{6}; \frac{6}{6}, \frac{0}{6}, \frac{0}{6}\right]$	$\left[\frac{6}{6}; \frac{6}{6}, \frac{0}{6}, \frac{0}{6}\right]$	$[3; 2, 1, 1]$	$\left[\frac{4}{5}; \frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right]$	$\left[\frac{5}{6}; \frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right]$
$[1; 1, 1, 0]$	$\left[\frac{3}{6}; \frac{3}{6}, \frac{3}{6}, \frac{0}{6}\right]$	$\left[\frac{3}{6}; \frac{3}{6}, \frac{3}{6}, \frac{0}{6}\right]$	$[2; 2, 1, 1]$	$\left[\frac{2}{5}; \frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right]$	$\left[\frac{2}{6}; \frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right]$

The Shapley-Shubik power vector, as well as the Banzhaf vector, of a simple game v coincides with that of its dual game v^d (Theorem 5 in Dubey and Shapley 1979). The result follows because v and v^d may also coincide. A weighted representation for the dual game can be obtained from a representation of the original game:

Lemma 3.7 *Let v be a weighted game with integer representation $(q; w_1, \ldots, w_n)$, and let $w(S) = \sum_{i \in S} w_i$, then*

$$(w(N) - q + 1; w_1, \ldots, w_n)$$

is an representation of its dual game v^d.

The four representation-compatible power indices studied in this paper are defined using two types of polytopes: *weight* polytope and *representation* polytope. To formally define the polytopes, let $\mathcal{W}^m$ be the set of minimal winning coalitions and $\mathcal{L}^m$ the set of maximal losing coalitions. The weight polytope is given by

$$\mathsf{V}(v) = \left\{ w \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^{n} w_i = 1,\, w(S) \geq w(T) \quad \forall S \in \mathcal{W}^m,\, T \in \mathcal{L}^m \right\}.$$

The representation polytope is given by

$$\mathsf{R}(v) = \left\{ (q, w) \in \mathbb{R}^{n+1}_{\geq 0} \mid \sum_{i=1}^{n} w_i = 1,\, w(S) \geq q \quad \forall S \in \mathcal{W}^m,\, w(T) \leq q \quad \forall T \in \mathcal{L}^m \right\}.$$

Let us illustrate the computation of average normalized weights on the example discussed in the introduction. The weight polytope $\mathsf{V} = \mathsf{V}(v)$ of the game $v = [51; 47, 46, 5, 2]$ is defined by the following system of inequalities

$$\begin{aligned}
&w_1 + w_2 \geq w_1 + w_4, \quad w_1 + w_2 \geq w_2 + w_4, \quad w_1 + w_2 \geq w_3 + w_4,\\
&w_1 + w_3 \geq w_1 + w_4, \quad w_1 + w_3 \geq w_2 + w_4, \quad w_1 + w_3 \geq w_3 + w_4,\\
&w_2 + w_3 \geq w_1 + w_4, \quad w_2 + w_3 \geq w_2 + w_4, \quad w_2 + w_3 \geq w_3 + w_4,
\end{aligned}$$

in addition to $w_1 + w_2 + w_3 + w_4 = 1$ and $w_i \geq 0$. Eliminating redundant inequalities yields

$$w_1 + w_2 \geq w_3 + w_4,\;\; w_1 + w_3 \geq w_2 + w_4,\;\; w_2 + w_3 \geq w_1 + w_4,\;\; w_4 \geq 0,\;\; w_1 + w_2 + w_3 + w_4 = 1.$$

The variables w_1, w_2, w_3 are symmetric. By assuming a specific ordering of these variables, we can decompose the integration domain V into six parts P, such that the resulting six integrals are equal. Moreover, it suffices to compute the average normalized weight for voter 4, because

$$\int_{\mathsf{V}} w_1 \,\mathrm{d}\,\mathsf{V} = \int_{\mathsf{V}} w_2 \,\mathrm{d}\,\mathsf{V} = \int_{\mathsf{V}} w_3 \,\mathrm{d}\,\mathsf{V}.$$

Let the ordering be $w_1 \geq w_2 \geq w_3$. Substituting $w_1 = 1 - w_2 - w_3 - w_4$ yields

$$\mathsf{P} = \left\{ (w_2, w_3, w_4) \in \mathbb{R}^3 \mid w_2 \geq w_3 \geq w_4 \geq 0,\, 2w_2 \geq 1 - 2w_3,\, 2w_2 \leq 1 - w_3 - w_4 \right\}.$$

To obtain the integration domain P, note that $\max\{w_4 \mid w \in \mathsf{V}\} = \frac{1}{4}$. Since $w_1 \geq w_2 \geq w_3 \geq w_4$ and $w_1 + w_2 + w_3 + w_4 = 1$, the maximum of w_3 given w_4 is $\frac{1-w_4}{3}$. Therefore,

$$\begin{aligned}
\int_{\mathsf{V}} w_4 \,\mathrm{d}\,\mathsf{V} &= 6 \int_{\mathsf{P}} w_4 \,\mathrm{d}\,\mathsf{P} = 6 \int_0^{\frac{1}{4}} \int_{w_4}^{(1-w_4)/3} \int_{\max(w_3, 1/2-w_3)}^{(1-w_3-w_4)/2} w_4 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4\\
&= 6 \int_0^{\frac{1}{4}} \int_{w_4}^{\frac{1}{4}} \int_{1/2-w_3}^{(1-w_3-w_4)/2} w_4 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4 +\\
&+ 6 \int_0^{\frac{1}{4}} \int_{\frac{1}{4}}^{(1-w_4)/3} \int_{w_3}^{(1-w_3-w_4)/2} w_4 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4 = \frac{1}{1536}.
\end{aligned}$$

The volume of V is given by integrating over 1:

$$\int_{\mathsf{V}} \mathrm{d}\,\mathsf{V} = \frac{1}{96}.$$

The average normalized weight of voter 4 thus equals $\frac{1}{16}$.

Replacing w_4 in the integrand by w_1, w_2, w_3 yields $\frac{19}{4608}$, $\frac{5}{2304}$ and $\frac{1}{288}$. By the symmetry of w_1, w_2, w_3,

$$\int_{\mathsf{V}} w_1 \,\mathrm{d}\,\mathsf{V} = \int_{\mathsf{V}} w_2 \,\mathrm{d}\,\mathsf{V} = \int_{\mathsf{V}} w_3 \,\mathrm{d}\,\mathsf{V} = 6\int_{\mathsf{P}} \frac{\overbrace{w_1 + w_2 + w_3}^{=1-w_4}}{3} \,\mathrm{d}\,\mathsf{P} = \frac{1}{3}\cdot\left(\frac{19}{4608} + \frac{1}{288} + \frac{5}{2304}\right) = \frac{5}{1536}.$$

This yields the following vector of average normalized feasible weights $\left(\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{16}\right)$.

We now consider the computation of the average representation based on the polytope R. Since $w_1 \geq w_2 \geq w_3$, a valid quota q must fulfill $w_1 + w_4 = 1 - w_2 + w_3 \leq q \leq w_2 + w_3$, so that

$$\mathsf{R} = \mathsf{R}(v) = \{(q, w) \mid w \in \mathsf{V},\, 1 - w_2 + w_3 \leq q \leq w_2 + w_3\}.$$

Following the above reasoning, we obtain

$$\begin{aligned}
\int_{\mathsf{R}} w_4 \,\mathrm{d}\,\mathsf{R} &= 6\int_0^{\frac{1}{4}} \int_{w_4}^{(1-w_4)/3} \int_{\max(w_3, 1/2-w_3)}^{(1-w_3-w_4)/2} \int_{1-w_2-w_3}^{w_2+w_3} w_4 \,\mathrm{d}\,q \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4 \\
&= 6\int_0^{\frac{1}{4}} \int_{w_4}^{\frac{1}{4}} \int_{1/2-w_3}^{(1-w_3-w_4)/2} \int_{1-w_2-w_3}^{w_2+w_3} w_4 \,\mathrm{d}\,q \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4 + \\
&+ 6\int_0^{\frac{1}{4}} \int_{\frac{1}{4}}^{(1-w_4)/3} \int_{w_3}^{(1-w_3-w_4)/2} \int_{1-w_2-w_3}^{w_2+w_3} w_4 \,\mathrm{d}\,q \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_3 \,\mathrm{d}\,w_4 = \frac{1}{23040}.
\end{aligned}$$

The volume of the polytope R equals $\frac{1}{1152}$. The average representation of the game reads $\left(\frac{1}{2}; \frac{19}{60}, \frac{19}{60}, \frac{19}{60}, \frac{1}{20}\right)$, where the average quota can be obtained by integrating over q.

The average normalized weights and the average representation come close to fulfilling the criteria for coherent measures of voting power provided in Definition 2.9. By construction, they are symmetric, positive, efficient and strongly monotonic according to Definition 2.12. Strong monotonicity in the sense of Isbell's desirability relation in Definition 2.10 follows, because $i \succ j$ implies $w_i > w_j$ in each representation of a given weighted game. However, they do not satisfy the null property, as this property was not accounted for in the underlying set of inequalities. Indeed, in the above example the fourth voter is a null, yet her weight in the vector of average weights power does not vanish.

To ensure coherency, we restrict the polytopes so that all nulls receive the value of zero. The null-revealing weight polytope is given by

$$\mathsf{V}^d(v) = \mathsf{V}(v) \cap \left\{w \in \mathbb{R}^n_{\geq 0} \mid w_i = 0 \quad \forall i \in D\right\}.$$

The null-revealing representation polytope is given by

$$\mathsf{R}^d(v) = \mathsf{R}(v) \cap \left\{ w \in \mathbb{R}^n_{\geq 0} \mid w_i = 0 \quad \forall i \in D \right\}.$$

From Lemma 3.4 we conclude that the $(t-1)$-dimensional volume of $\mathsf{V}^d(v)$ and the t-dimensional volume of $\mathsf{R}^d(v)$ is non-zero for each weighted game v, where $1 \leq t \leq n$ denotes the number of non-null voters of v. The following definition based on the restricted polytopes $\mathsf{V}^d(v)$ and $\mathsf{R}^d(v)$ has been proposed in Kaniovski and Kurz (2015):

Definition 3.8 *The average weight index of voter i in a weighted game v is given by*

$$AWI_i(v) = \frac{\int_{\mathsf{V}^d} w_i \,\mathrm{d}\, w}{\int_{\mathsf{V}^d} \mathrm{d}\, w}.$$

Similarly, the average representation index of voter i in a weighted game v is given by

$$ARI_i(v) = \frac{\int_{\mathsf{R}^d} w_i \,\mathrm{d}(q, w)}{\int_{\mathsf{R}^d} \mathrm{d}(q, w)}.$$

In the above definition, all integrals are understood as multiple integrals.

The AWI and ARI indices have a clear geometric interpretation as *centroids* of their respective polytopes. Since the centroid of a convex polytope belongs to its interior, the power vectors lie in the interiors of their respective polytopes and are therefore representation-compatible. This simple observation furnishes the defining property of the new indices.

It is important to note that we can safely remove the nulls players and the null-related restrictions prior to computing the indices. In fact, the more null voters a game has, the simpler the power computations are. The validity of this procedure follows from a result, which also holds for the Banzhaf index. Given a weighted game $v : 2^N \to \{0, 1\}$ with the set of null voters $D \subset N$, we define the null-reduced game $v' : 2^{N \backslash D} \to \{0, 1\}$ via $v'(T) = v(T)$ for all $T \subseteq N \backslash D$. All nulls receive the value of zero in the outcome vector.

Lemma 3.9 *Given a sequence of power indices $g^n : \mathcal{S}_n \to \mathbb{R}^n$ for all $n \in \mathbb{N}$, let $\tilde{g}^n : \mathcal{S}_n \to \mathbb{R}^n$ be defined via $\tilde{g}^n_i(v) = g^m_i(v')$ for all non-nulls i and by $\tilde{g}^n_j(v) = 0$ for all nulls j, where m is the number of non-nulls in v and v' arises from v by removing the nulls. The power index $\tilde{g}^n$ now satisfies the null property.*

We call $\tilde{g}^n$ the null-revealing version of a given sequence of power indices g^n. The above lemma shows that the presence of nulls reduces the dimension of the polytopes, thus simplifying computations.

Tables A.1 and A.2 of Appendix A.3 list power distributions according to the AWI and ARI for all weighted games with up to five voters. Power distributions in games with fewer than five voters can be obtained from games in which the additional voters are assumed to be nulls. For example, the power distribution in the game $[3; 2, 1, 1]$, in which none of the three voters is a null, is given by the first three coordinates of the power vector for the game $[3; 2, 1, 1, 0, 0]$, in which the additional two voters are nulls. This holds for each of the four power indices.

The AWI and ARI preserve the types of voters implied in the equivalence relations of Definition 2.3. We call such indices *type-revealing*.[4] However, the required computational burden

[4] Note that any strongly monotonic and symmetric power index is type-revealing.

can be reduced in the cases where not every player forms its own equivalence class of voters. Imposing the following additional restrictions on the polytopes leads to a new pair of representation-compatible power indices that are computationally less demanding than AWI and ARI.

$$\begin{aligned}\mathsf{V}^t(v) &= \mathsf{V}^d(v) \cap \left\{w \in \mathbb{R}^n_{\geq 0} \mid w_i = w_j \quad \forall i,j \in N \text{ s.t. } i \sim j\right\},\\ \mathsf{R}^t(v) &= \mathsf{R}^d(v) \cap \left\{w \in \mathbb{R}^n_{\geq 0} \mid w_i = w_j \quad \forall i,j \in N \text{ s.t. } i \sim j\right\}.\end{aligned}$$

Lemma 3.4 implies that $(t-1)$-dimensional volume of $\mathsf{V}^t(v)$ and the t-dimensional volume of $\mathsf{R}^t(v)$ is non-zero for each weighted game v, where $1 \leq t \leq n$ denotes the number of equivalence classes of voters of v, excluding the null voters (who always form a separate class). The case of $t = n$ can be handled separately, as in this case all voters are by definition equally powerful.

Definition 3.10 *The average weight preserving types index of voter i in a weighted game v is given by*

$$AWTI_i(v) = \frac{\int_{\mathsf{V}^t} w_i \,\mathrm{d}\, w}{\int_{\mathsf{V}^d} \mathrm{d}\, w}.$$

Similarly, the average representation preserving types index of voter i in a weighted game v is given by

$$ARTI_i(v) = \frac{\int_{\mathsf{R}^t} w_i \,\mathrm{d}(q,w)}{\int_{\mathsf{R}^d} \mathrm{d}(q,w)}.$$

The computation of AWTI and ARTI follows the same procedures described the example above, except that it uses the restricted versions of the polytopes instead of their unrestricted counterparts. A complete example of the above calculations is provided in Appendix A.1. Note that the dimension of the polytopes for the new indices are typically smaller than for the AWI and ARI. Moreover, each equivalence class of voters contributes a single integral, since the weights of the players within the same equivalence class are assumed to be equal. These two factors reduce the computational burden.

Tables A.3 and A.4 of Appendix A.3 list AWTI and ARTI indices for all weighted games with up to four voters. We conclude the presentation of the power indices with a remark on duality (Definition 3.6).

Lemma 3.11 *The average weight index (AWI) and the average representation index (ARI) coincide for the pairs of a weighted game v and its dual v^d.*

Proof. According to Lemma 3.7, the integer representations of v and v^d are in bijection. Let $(q, w_1, \ldots, w_n)$ be a normalized representation of v, then $(1 - q + \varepsilon, w_1, \ldots, w_n)$ is a normalized representation of v^d for a sufficiently small $\varepsilon > 0$, as $q \in (0, 1]$. If we require that the weight of each winning coalition in the dual game strictly exceed the quota, then we can choose $1 - q$ as a quota for the dual game, while also retaining the weights. In view of Lemma 3.4, this difference between a strict and non-strict inequality can be neglected when computing the indices, which proves the lemma. □

4 Computational complexity

The computation of the new power indices involves integration over full-dimensional convex polytopes with rational vertices. Finding the power distribution among n non-null voters according to any of the four indices requires evaluating n integrals: $n-1$ with the integrands x_i for each $i = 1, 2, \ldots, (n-1)$, and one with the integrand 1 for the volume of the polytope. This is the worst-case scenario for the AWTI and ARTI. In practice, the number of integrals involved in computing the AWTI and ARTI may be significantly lower, because these indices require all voters belonging to the same equivalence class to be equally powerful. This is likely to reduce the dimension of the problem, because the number of equivalence classes is typically smaller than the number of voters, reducing the number of integrals to be evaluated. For any index, all integrals can be evaluated in parallel and the power the last voter can obtained by subtracting the normalized sum of the integrals from 1.

Power computations presented in this paper were performed using the software `LattE` by Baldoni, Berline, De Loera, Dutra, Köppe, Moreinis, Pinto, Vergne and Wu (2014). The theoretical background behind the methods used in `LattE` are detailed in De Loera, Dutra, Köppe, Moreinis, Pinto and Wu (2013). `LattE` decomposes a polytope with rational vertices in either a union of disjoint simplices (triangulation), or signed cones. Triangulation is faster than cone decomposition for $n \leq 7$, but becomes unfeasible starting from $n = 10$. The cone decomposition remains feasible for moderate values of n. The rationality of the vertices poses no limitations, because one can specify any weighted voting game using integer weights, for example, the minimum weight representation, thereby ensuring rationality of the vertices. The general problem of finding the volume of a convex polytope and the problem of computing its centroid both belongs to the complexity class #P. The practical feasibility and the computational effectiveness of the two decomposition methods largely depend on the geometry of the polytope implied in a specific weighted voting game.

Table 2 provides examples of times to compute the four indices, averaged over 100 randomly generated (proper) weighted voting games. The presence of equivalence classes comprising more than one member significantly reduces the computational burden of the AWTI and ARTI indices relative to the AWI and ARI indices. Although all power computations in this paper are exact, computing the power distribution in games with sufficiently many players will need to rely on numerical integration.

Table 2: Seconds to compute.

n	3	4	5	6	7	8	9	10
AWI	0.01	0.02	0.04	0.23	2.55	23.68	275.42	10726.21
AWTI	0	0	0	0.01	0.05	0.36	1.34	3.82
ARI	0.01	0.02	0.04	0.10	0.41	5.19	72.32	10307.38
ARTI	0	0.01	0.01	0.04	0.08	0.47	1.12	2.74

Averaged over 100 weighted voting games. Each index was computed on a single Intel Xeon E5-2640v3 2.60Ghz core.

5 The properties of representation-compatible power indices

The common criteria for choosing an index include the existence of a game-theoretic axiomatizing, consistency with certain stochastic models of voting or immunity to certain voting paradoxes. Table 3 compares the four power indices (AWI, ARI, AWTI, ARTI) to several existing power indices. Some of these power indices are well-known, whereas others have only recently been introduced. All indices introduced in this paper are coherent measures of power; they satisfy **Null**, **Eff**, **Invar** and **Str.Mon** (Definition 2.9). These properties follow by construction of the polytopes. Most researchers agree that a power index should at least be coherent. Yet two well-known power indices by Deegan and Packel (1978) and Holler (1982) violate monotonicity, and are therefore not coherent.

Table 3: Basic properties and immunities to voting paradoxes.

Index	**Null**	**Eff**	**Invar**	**Str.Mon**	**Prop**	**Type-Rev**	**Bloc**	**Don**	**Bic.Meet**
Shapley and Shubik (1954)	✓	✓	✓	✓		✓	✓	✓	
Banzhaf (1965)	✓	✓	✓	✓		✓			✓
Johnston (1978)	✓	✓	✓	✓		✓			
Deegan and Packel (1978)	✓	✓	✓						
Holler (1982)	✓	✓	✓						
Freixas and Kaniovski (2014)	✓	✓	✓	✓	✓	✓			✓
AWI	✓	✓	✓	✓	✓	✓			
ARI	✓	✓	✓	✓	✓	✓			
AWTI	✓	✓	✓	✓	✓	✓			
ARTI	✓	✓	✓	✓	✓	✓			

The defining property of the four indices is representation-compatibility, which ensures proportionality (**Prop**) between power and weight. This property follows from the fact that the centroid of a convex polytope belongs to its interior. Proportionality between power and weight makes representation-compatible indices convenient measures of power. The MSR Index introduced in Freixas and Kaniovski (2014) is the only existing power index that has this property. The four indices are also type-revealing (**Typ.Rev**).

5.1 Distributing parliamentary seats: an example

To illustrate this convenience, suppose we wish to fill the Austrian parliament (Nationalrat) following the general election of 2013. Six parties have attained the electoral threshold of 4 percent required to secure a seat in parliament. Their popular votes are listed in the first column of Table 4. The Austrian parliament uses the D'Hondt method to allocate 183 seats among the political parties that passed the threshold. The actual seat distribution is given in the second column.

Despite the fact that the D'Hondt method is not based on power computations, the resulting distribution of voting power in the parliament resembles the distribution of power implied in the popular vote. This occurs because the D'Hondt method tries to achieve proportionality, thus preserving the game representation implied in the popular vote. In our example, the resemblance is complete. For example, under plurality voting rule we have the following weighted voting games based on popular votes and parliamentary seats, respectively: $[2215538; 1258605, 1125876, 962313, 582657, 268679, 232946]$ and $[92; 52, 47, 40, 24, 11, 9]$. These games have identical power distributions according to the Shapley-Shubik index (SSI), as the third column of Table 4 shows.

Suppose that, instead of using the D'Hondt method, we allocated the parliamentary seats according to the distribution of power implied in the popular vote under plurality voting rule. For example, we could distribute the seats according to power distributions obtained from the SSI or AWI, with the implied seat distributions provided in Table 4. The distribution of seats according to the SSI index implies a different power distribution than the one given in the third column, as power vectors of the games $[92; 52, 47, 40, 24, 11, 9]$ and $[92; 67, 49, 49, 6, 6, 6]$ differ. On the contrary, $[92; 52, 47, 40, 24, 11, 9]$ and $[92; 63, 44, 44, 11, 11, 11]$ have identical power vectors according to the AWI. A seat distribution according to the AWI allows us to easily discern the power distribution from the weight distribution, because the AWI power vector is a representation of the game.

The last two columns provide the power and seat distributions according to the AWTI. In this particular example, the distributions implied by AWI and AWTI are very similar. The weighted voting game has three equivalence classes of voters:

$$\{\text{SPÖ}, \{\text{ÖVP, FPÖ}\}, \{\text{Green, Team Stronach, NEOS}\}\}.$$

The AWTI explicitly imposes the equality of power among the equivalent voters, although in this example all other indices also respect the equivalence classes.

Table 4: Austrian Nationalrat election, 2013.

	Popular Votes	Seats	SSI	SSI Seats	AWI	AWI Seats	AWTI	AWTI Seats
SPÖ	1,258,605	52	0.367	67	0.342	63	0.350	64
ÖVP	1,125,876	47	0.267	49	0.242	44	0.233	43
FPÖ	962,313	40	0.267	49	0.242	44	0.233	43
Green	582,657	24	0.033	6	0.058	11	0.061	11
Team Stronach	268,679	11	0.033	6	0.058	11	0.061	11
NEOS	232,946	9	0.033	6	0.058	11	0.061	11
Quota	2,215,538	92		92		92		92

The total number of parliamentary seats according to the AWI is 184 not 183. This rounding error can be rectified by subtracting one seat from the largest party, as this would leave the power distribution unchanged according to AWI.

The distribution of seats based on the new indices uses the representation provided by the popular votes as a template. A different problem is that of designing a weighted voting game with an arbitrary given power distribution - a problem of practical importance for institutional design. Although this inverse problem may not have an exact solution, an approximate solution for a representation-compatible power index can easily be found using a grid search for a quota that minimizes an objective function, say the sum of squared deviations between weights and powers. The desired power distribution becomes the weight distribution in the solution, appropriately rescaled should integer-valued voting weights be needed. This stands in contrast to the classical power indices, whose inverse problems are significantly more difficult.[5]

5.2 A comparison with the classical power indices

The above example shows that the SSI is not representation-compatible in games with more than three voters, and neither is the BZI. If a power vector is not representation-compatible, then it must lie outside the polytope containing the feasible weights. To get a broad picture on how representation-compatible power indices differ from the classical indices by Banzhaf (BZI) and Shapley-Shubik (SSI), for each game we compute the Euclidean distance between the six measures, and consider the distribution of the distances for all games of a given size.

Appendix A.2 contains the boxplots of the distances for all games with sizes up to a given n. Similarly to Tables in Appendix A.3, the games differ in their partitions in the equivalence sets, and are defined in terms of the minimum sum representations. The bar in the middle shows the median distance. The top whisker ranges from the 99% quantile to the 75% quantile. The

[5]See, for example, the fixed-point iteration methods for obtaining the inverse solution for the Banzhaf index in Aziz, Paterson and Leech (2007).

bottom whisker ranges from the 25% quantile to the 1% quantile. The box thus covers the range of 25-75%.

The differences between the classical and representation-compatible indices become apparent as n increases. The bottom panels suggest that representation-compatible indices lie closer to each other than the BZI and SSI, the former appears to lie closer to representation-compatible indices than the latter. The median distance between the AWI and the SSI is slightly larger than the median distance between the BZI and the AWI. This may suggest that the BZI is more likely to be representation-compatible than the SSI. But a power index is representation-compatible if it lies in the interior of the null-revealing polytope $\mathsf{V}^d(v)$, whose centroid is the AWI power vector. Being closer to the centroid does not imply being closer to the boundary of the polytope.

Among the four representation-compatible indices, the AWI and ARI lie closer to each other then the AWTI and ARTI. The distances between the representation-compatible indices decreases with an increasing n, which is not surprising given that polytopes containing representation-compatible power distributions are likely to shrink as n increases.

5.3 Immunities to voting paradoxes

A voting paradox occurs when a power measure behaves counter-intuitively. The study of voting paradoxes is of longstanding interest in the literature on voting power, because understanding why a paradox occurs yields insight about the properties of a measure. Many voting paradoxes have been discussed in Brams (1990) and Felsenthal and Machover (1998). Some of the paradoxes are mere oddities, others can be a serious impediment to the application of a measure liable to it.

Table 5: Block paradox in game [37;25,20,17,15,9,6,2,1].

Weight	25	20	17	15	9	6	2	1
BZI	0.274	0.226	0.188	0.168	0.063	0.053	0.0240	0.005
SSI	0.287	0.230	0.196	0.163	0.054	0.046	0.0202	0.004
MSRI	0.262	0.213	0.180	0.148	0.082	0.066	0.0328	0.016
AWI	0.267	0.226	0.196	0.140	0.082	0.056	0.0283	0.006
ARI	0.266	0.224	0.194	0.140	0.082	0.057	0.0288	0.007
AWTI	0.267	0.226	0.196	0.140	0.082	0.056	0.0283	0.006
ARTI	0.266	0.224	0.194	0.140	0.082	0.057	0.0288	0.007
Weight	25	20	17	15	9	6	3	0
BZI	0.282	0.223	0.185	0.165	0.068	0.049	0.0291	0
SSI	0.293	0.226	0.193	0.160	0.060	0.043	0.0262	0
MSRI	0.273	0.212	0.182	0.152	0.091	0.061	0.0303	0
AWI	0.272	0.225	0.197	0.140	0.087	0.051	0.0281	0
ARI	0.272	0.224	0.195	0.141	0.087	0.052	0.0284	0
AWTI	0.272	0.225	0.197	0.140	0.087	0.051	0.0281	0
ARTI	0.272	0.224	0.195	0.141	0.087	0.052	0.0284	0

Felsenthal and Machover (1998) identify three voting paradoxes to which any reasonable measure of power should not be liable. These are the bloc (**Bloc**), donation (**Don**) and bicameral meet (**Bic.Meet**) paradoxes. In the following, we provide examples showing that all of representation-compatible indices are liable to these paradoxes, so they have nothing to recommend in this department. But since none of the existing indices have all the required immunities, the question of which index to use cannot be settled based on immunity to paradoxes alone.

Bloc paradox: Respecting the bloc postulate means that if two or more voters form a bloc by adding their votes, the power of the bloc should not be lesser than the power of either voter alone. Table 5 provides an example of a game, in which the smallest two voters form a bloc by joining their voting weights, and lose power as a result, if only slightly. The BZI and the SSI do not show the paradox in this example, although examples are known in which the BZI is liable to the bloc paradox. Also, the MSRI of Freixas and Kaniovski (2014) displays the bloc paradox in this example.

Donation paradox: Respecting donation means that if one voter gives some of her votes to another, the power of the donor should not increase as a result. Felsenthal and Machover (1998) provide examples in which the Banzhaf and Johnston indices show both bloc and donation paradoxes. Freixas and Molinero (2010) study the frequency of the occurrence of the donation paradox in weighted games with a small number of players, providing examples for the Banzhaf and Johnston indices. The Shapley-Shubik index is immune to both the bloc and donation paradoxes. Freixas and Kaniovski (2014) provide an example, which also shows that the MSR index is liable to the donation paradox.

The example in Table 6 shows that representation-compatible indices are liable to the donation paradox. In the game $[13; 9, 4, 3, 2, 1]$, the largest voter gains power by donating one vote to the second largest voter according to the Shapley-Shubik index, but gains power according to all other indices. In example, the BZI and the MSRI shows the donation paradox. The Shapley-Shubik index is immune to both the bloc and donation paradoxes.

Table 6: Donation paradox in game [13;9,4,3,2,1].

Weight	9	4	3	2	1
BZI	0.524	0.238	0.143	0.048	0.048
SSI	0.617	0.200	0.117	0.033	0.033
MSRI	0.417	0.250	0.167	0.083	0.083
AWI	0.518	0.247	0.138	0.048	0.048
ARI	0.501	0.247	0.143	0.054	0.054
AWTI	0.548	0.258	0.123	0.035	0.035
ARTI	0.522	0.257	0.132	0.045	0.045
Weight	8	5	3	2	1
BZI	0.500	0.300	0.100	0.100	0
SSI	0.583	0.250	0.083	0.083	0
MSRI	0.429	0.286	0.143	0.143	0
AWI	0.535	0.270	0.098	0.098	0
ARI	0.513	0.273	0.107	0.107	0
AWTI	0.602	0.249	0.075	0.075	0
ARTI	0.558	0.258	0.092	0.092	0

The bicameral meet: An index of power respects bicameral meet if the ratio of powers of any two voters belonging to the same assembly prior to a merge with a different assembly is preserved in the joint assembly. This property is useful when measuring the voting power of shareholders, because the relative powers of shareholders comprising a minority voting assembly with a total voting weight, represented by their joint holdings, carries over to the grand voting assembly, represented by the total worth of the company.

The bicameral meet of two simple voting games $(N_1, \mathcal{W}_1)$ and $(N_2, \mathcal{W}_2)$ is a simple voting game $(N, \mathcal{W})$, with an assembly $N = N_1 \cup N_2$, and a set of winning coalitions $\mathcal{W} = \{S \subseteq N : S = S_1 \cup S_2, S_1 \in \mathcal{W}_1, S_2 \in \mathcal{W}_2\}$. The two assemblies have no voters in common, so $N_1 \cap N_2 = \emptyset$. The bicameral meet postulate requires that if i and j are non-null voters in a game $(N_1, \mathcal{W}_1)$, then the ratio of power of voter i to the power of voter j in the joint game $(N, \mathcal{W})$ should be equal to the ratio of their powers in the original game $(N_1, \mathcal{W}_1)$.

Table 7: Added blocker paradox in game $[3; 2, 1, 1] \cup [5; 5] = [8; 2, 1, 1, 5]$.

Weight	2	1	1		Voter 1 / Voter 2
BZI	0.600	0.200	0.200		3
SSI	0.667	0.167	0.167		4
MSRI	0.500	0.250	0.250		2
AWI	0.611	0.194	0.194		3.143
ARI	0.583	0.208	0.208		2.8
AWTI	0.667	0.167	0.167		4
ARTI	0.611	0.194	0.194		3.143
Weight	2	1	1	5	Voter 1 / Voter 2
BZI	0.375	0.125	0.125	0.375	3
SSI	0.417	0.083	0.083	0.417	5
MSRI	0.333	0.167	0.167	0.333	2
AWI	0.396	0.104	0.104	0.396	3.8
ARI	0.383	0.117	0.117	0.383	3.286
AWTI	0.375	0.125	0.125	0.375	3
ARTI	0.361	0.139	0.139	0.361	2.6

Freixas and Kaniovski (2014) prove that a bicameral meet of two complete games is complete if at least one of the two constituent games has only one minimum winning coalition. A special case of the bicameral meet postulate is the added blocker postulate, which says that adding a vetoer (Definition 2.5) to a weighted game should not change the ratio of powers of any two incumbent voters. If an index is liable to an added blocker paradox, it also fails the bicameral meet postulate, see Felsenthal and Machover (1998) (p. 270). In Table 8, we use their example to show that neither of representation-compatible indices satisfies the bicameral meet postulate. The example involves adding a blocker with a weight of 5 to the game $[3; 2, 1, 1]$, and adjusting the quota in such a way that the set of minimal winning coalitions of the joint games equals the union of the sets of minimal winning coalitions in each game. This amounts to joining the games $[3; 2, 1, 1]$ and $[5; 5]$. Note that the blocker is a dictator in the added game. Since the second game has a single coalition, which is trivially minimal winning, the joint game is complete. The bicameral meet postulate does not hold for representation-compatible indices, because adding the blocker changes the power ratios of the players. This postulate is satisfied by the BZI and the MSRI.

There are many lesser paradoxes and other properties that may distinguish between power indices. One useful property is neutrality in symmetric voting games. In a symmetric weighted voting game, each player commands an equal number of votes. For a power measure to respect neutrality, the power of a voting bloc must equal the sum of individual powers of its members, so that satisfying the bloc postulate does not carry strategic implications. The power vectors for games $[3;1,1,1,1]$ and $[3;2,1,1]$ in Tables of Appendix A.3 clearly show that representation-compatible indices do not respect neutrality. This property is satisfied by the MSRI.

6 Integral weights and type preservation

A normalization of voting weights is unreasonable if they represent the number of shares of a corporation or the number of members of a political party. In these cases, we require the weights to be integers. This observation has led to the development of a power index based on the minimum sum integer representations, called the MSR index (Freixas and Kaniovski 2014).

Consider the weighted game $v = [2;1,1,1]$, for which there exist 1176 feasible integer weight vectors with the total weight of 100. The average of all these vectors equals $\left(\frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right)$, yielding $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as the average weight distribution, which is not surprising, given the inherent symmetry of the game.

Things get more interesting if we consider the game $v = [3;2,1,1]$. Table 8 lists the number of feasible integer weight vectors for an increasing total weight, as well as the average weight distributions. The distribution appears to converge to $\left(\frac{11}{18}, \frac{7}{36}, \frac{7}{36}\right)$, which equals the AWI for this game. This convergence can be rigorously established by numerically approximating the integrals in Definition 3.8 over a successively finer equally spaced grid inside the polytope. A similar result holds if an integer-valued quota is taken into account, in which case we obtain the ARI in the limit. The null-revealing and type-revealing property is also preserved in the limit.

To obtain power indices based on integer representations, we can minimize the sum of weights instead of taking it to infinity. Unfortunately, the minimum sum representations are not unique for $n \geq 8$ (Kurz 2012). Nevertheless, one can take a convex combination of all such minimum sum representations, which yields, after a normalization, the MSR index recently introduced in Freixas and Kaniovski (2014).[6] One motivation for minimizing weights is to minimizing the cost of political representation by minimizing the number of representatives. Another motivation for minimum sum representation and the MSR index is given in Ansolabehere, Snyder, Strauss and Ting (2005), who argue that many observations on the formation of coalition governments are more consistent with minimal integer-voting weights, rather than power distributions implied by the classical power indices.

The minimum sum integer representations are null-revealing, but not type-revealing. Take the two representations $(12,7,6,6,4,4,4,3,2)$ and $(12,7,6,6,4,4,4,2,3)$, which are minimum sum representations of the same game. Indeed, there exists no integer representation of this game with a weight sum smaller than 36. Both representations are not type-revealing since the equivalence classes of voters are given by $\{1\}$, $\{2,3\}$, $\{4,5,6\}$, and $\{7,8\}$, while $w_7 \neq w_8$. Implementing one of the two representations may cause some confusion, as one might erroneously think that one of the two voters is more powerful than the other. The unique type-revealing minimum sum representation for this example is given by $(14;8,7,7,5,5,5,3,3)$. It has a total weight of 43 instead of 36. Unfortunately, even the type-revealing minimum sum representation can be non-unique if at least $n = 9$ voters are involved, see Kurz (2012). At the very least, one can define a minimum sum representation type-revealing index of a weighted game as a normalization of the convex combination of the corresponding set of type-revealing minimum sum representations. Some uniqueness results for both the minimum sum representation and the type-revealing minimum sum representation exist for special classes of weighted games. Freixas

[6]Minimizing $q + \sum_{i=1}^{n} w_i$ instead of $\sum_{i=1}^{n} w_i$ makes no difference.

and Kurz (2014) proved that weighted games with up to two equivalence classes of voters admit a unique minimum sum representation. For minimum sum representations, the authors give non-unique examples for four equivalence classes of voters and conjecture a uniqueness result for three equivalence classes of voters.

For the algorithmic aspects of computing minimum sum representations and the MSR index, see Kurz (2012). The gist of this research is that the minimum sum representation can often be computed by solving a short sequence of linear programs. The computation of type-revealing minimum sum representations requires only minor adjustments.

Table 8: Convergence of feasible integer weights for $[3; 2, 1, 1]$.

Total weight	Number of integer representations	Weight distribution
100	1601	(0.608832,0.195584,0.195584)
1000	166001	(0.610888,0.194556,0.194556)
10000	16660001	(0.611089,0.194456,0.194456)
100000	1666600001	(0.611109,0.194446,0.194446)

7 Concluding remarks

The average representations of a weighted voting game can be used to obtain four representation-compatible indices of voting power for this type of voting game. The average representations are computed from weight and representation polytopes defined by the set of winning and losing coalitions of the game. The weight polytope is based on normalized voting weights, whereas the representation polytope also includes the quota.

These average representations come remarkably close to fulfilling the standard criteria for a coherent measure of voting power. They are symmetric, positive, efficient and strongly monotonic. But they do not respect the null property that assigns zero power to powerless players. This shortcoming is easily rectified by further restricting the polytopes. The resulting restricted average representations respect the null property and are coherent measures of power. The above modification suggests that we can endow the indices with qualities by tailoring the polytope. Explicit restrictions based on the equivalence classes of voters defined by the Isbell desirability relation lead to another pair of power indices that are, in many cases, computationally cheaper.

The defining property of the indices is representation-compatibility, which ensures proportionality between power and weight. By redistributing weights among the voters, we can redesign any given weighted voting game in such a way that the distribution of voting weight will also be the distribution of voting power. This allows us to read power directly from the weights, a convenient property that recommends representation-compatible indices as optimal representations for weighted voting games, or optimal designs for voting institutions. The obvious disadvantage is the computational intensity of integrating monomials on highly-dimensional polytopes, but the two indices introduced in this paper can alleviate this problem to some extent. Obtaining power distributions in weighted games with many players may require numerical integration based on random sampling.

Reflecting on the place representation-compatible indices may take among the existing measures of power, we believe that proportionality makes them ideal measures of power for voting institutions, in which the votes are distributed to the voter based on their contribution to a fixed purse. In this setting, voting power reflects the extent of a voter's control of the distribution of a fixed purse - the ultimate outcome of voting, measured by that voter's expected share in the purse.[7] If a voter's expected share of spoils coincides with the voter's contribution to the fixed purse, an equilibrium emerges in which voters will not wish to redistribute votes. This leads to a stable institutional design of vote-for-money institutions, such as a corporation.

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[7]Felsenthal and Machover (2004) refer to this notion of power as P-power.

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A Appendix

A.1 Example $[3; 2, 1, 1]$

The sets of minimal winning and maximal losing coalitions for the game $[3; 2, 1, 1]$ are, respectively, $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ and $\{\{1\}, \{2, 3\}\}$. Since voters 2 and 3 are equivalent, there are two equivalence classes in this game. There are no nulls.

Using Lemma 3.2, we obtain the following constraints:

$$\begin{aligned} w_1 + w_2 > w_1 &\iff w_2 > 0; \\ w_1 + w_3 > w_1 &\iff w_3 > 0; \\ w_1 + w_2 > w_2 + w_3 &\iff w_1 > w_3; \\ w_1 + w_3 > w_2 + w_3 &\iff w_1 > w_2. \end{aligned}$$

In addition, $w_1, w_2, w_3 \geq 0$ and $w_1 + w_2 + w_3 = 1$. Eliminating w_3 and removing the redundant constraints yields the following inequalities: $w_2 > 0$, $w_2 < 1 - w_1$, $w_2 > 1 - 2w_1$, $w_2 < w_1$. Since $1 - 2w_1 < w_1$ and $1 - w_1 > 0$, we have $w_1 \in \left(\frac{1}{3}, 1\right)$. For $w_1 \in \left(\frac{1}{3}, \frac{1}{2}\right)$, we have $w_2 \in \left(1 - 2w_1, w_1\right)$. For $w_1 \in \left[\frac{1}{2}, 1\right)$, we have $w_2 \in \left(0, 1 - w_1\right)$. The polytope is thus given by

$$\mathsf{V}^d(v) = \left\{(w_1, w_2) \in \mathbb{R}^2_{\geq 0} \mid w_2 \geq 0, w_2 \leq 1 - w_1, w_2 \geq 1 - 2w_1, w_2 \leq w_1\right\}.$$

Since there are no nulls in this game, the null-revealing polytope $\mathsf{V}^d(v)$ coincides with its non-revealing counterpart $\mathsf{V}(v)$.

For voter 1, we have

$$\iint\limits_{\mathsf{V}^d} w_1 \,\mathrm{d}\,w_1 \,\mathrm{d}\,w_2 = \int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} w_1 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} w_1 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_1 = \frac{1}{54} + \frac{1}{12} = \frac{11}{108}.$$

For voter 2, we obtain

$$\iint\limits_{\mathsf{V}^d} w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,w_2 = \int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} w_2 \,\mathrm{d}\,w_1 \,\mathrm{d}\,w_2 + \int_{\frac{1}{2}}^{1} \mathrm{d}\,w_1 \int_{0}^{1-w_1} w_2 \,\mathrm{d}\,w_2 \,\mathrm{d}\,w_1 = \frac{1}{48} + \frac{5}{432} = \frac{7}{216}.$$

The total volume of the polytope is given by

$$\iint\limits_{\mathsf{V}^d} \mathrm{d}\,w_1 \,\mathrm{d}\,w_2 = \int_{\frac{1}{3}}^{\frac{1}{2}} \int_{1-2w_1}^{w_1} \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{1-w_1} \mathrm{d}\,w_2 \,\mathrm{d}\,w_1 = \frac{1}{8} + \frac{1}{24} = \frac{1}{6},$$

This yields the following vector of average (normalized) feasible weights $\left(\frac{11}{18}, \frac{7}{36}, \frac{7}{36}\right)$.

The polytope for the average representation defined by Lemma 3.3 is given by

$$\mathsf{R}^d(v) = \left\{(q, w_1, w_2) \in \mathbb{R}^3_{\geq 0} \mid w_1 + w_2 \geq q, w_1 \leq q, 1 - w_1 \leq q, 1 - w_2 \geq q\right\},$$

since need to take into account the inequalities from the minimal winning and the maximal losing coalitions only, due to $w_1, w_2 \geq 0$ and $w_3 = 1 - w_1 - w_2 \geq 1 - w_2 - q \geq 0$. From $w_1 \leq q$ and $1 - w_1 \leq q$ follows $\frac{1}{2} \leq q \leq 1$. We can rewrite the four inequalities to $1 - q \leq w_1 \leq q$, with

$1-q \leq q$ for all $q \geq \frac{1}{2}$, and $q - w_1 \leq w_2 \leq 1-q$, with $q - w_1 \leq 1-q$ iff $w_1 \geq 2q-1$. Since $2q-1 \geq 1-q$ iff $q \geq \frac{2}{3}$ and $2q-1 \leq q$ for all $q \leq 1$, we have,

$$\begin{aligned}
\iint\limits_{\mathsf{R}^d} w_1 \,\mathrm{d}\, w_1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, q &= \int_{\frac{1}{2}}^{\frac{2}{3}} \mathrm{d}\, q \int_{1-q}^{q} w_1 \,\mathrm{d}\, w_1 \int_{q-w_1}^{1-q} \mathrm{d}\, w_2 + \int_{\frac{2}{3}}^{1} \mathrm{d}\, q \int_{2q-1}^{q} w_1 \,\mathrm{d}\, w_1 \int_{q-w_1}^{1-q} \mathrm{d}\, w_2 \\
&= \frac{31}{7776} + \frac{1}{243} = \frac{7}{864} \\
\iint\limits_{\mathsf{R}^d} w_2 \,\mathrm{d}\, w_1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, q &= \int_{\frac{1}{2}}^{\frac{2}{3}} \mathrm{d}\, q \int_{1-q}^{q} \mathrm{d}\, w_1 \int_{q-w_1}^{1-q} w_2 \,\mathrm{d}\, w_2 + \int_{\frac{2}{3}}^{1} \mathrm{d}\, q \int_{2q-1}^{q} \mathrm{d}\, w_1 \int_{q-w_1}^{1-q} w_2 \,\mathrm{d}\, w_2 \\
&= \frac{29}{15552} + \frac{1}{972} = \frac{5}{1728}, \\
\iint\limits_{\mathsf{R}^d} \mathrm{d}\, w_1 \,\mathrm{d}\, w_2 \,\mathrm{d}\, q &= \int_{\frac{1}{2}}^{\frac{2}{3}} \mathrm{d}\, q \int_{1-q}^{q} \mathrm{d}\, w_1 \int_{q-w_1}^{1-q} \mathrm{d}\, w_2 + \int_{\frac{2}{3}}^{1} \mathrm{d}\, q \int_{2q-1}^{q} \mathrm{d}\, w_1 \int_{q-w_1}^{1-q} \mathrm{d}\, w_2 \\
&= \frac{5}{648} + \frac{1}{162} = \frac{1}{72},
\end{aligned}$$

so that the average representation is given by $\left(\frac{7}{12}, \frac{5}{24}, \frac{5}{24}\right)$.

We now move from individual voting weights to weights aggregated by equivalence classes, as if voters belonging to the same class form a voting bloc with weight being equal to the sum of weights of its members.

The game has two classes: class A comprises voter 1, whereas voters 2 and 3 form class B. Let w_a be the voting weight of class A, which equals the weight of the first voter $w_a = w_1$. The AWTI polytope degenerates to an interval $\mathsf{V}^t(v) = \{w_a \in \mathbb{R}_{\geq 0} \mid 3w_a \geq 1, w_a \leq 1\}$. We have,

$$\int\limits_{\mathsf{V}^t} w_a \,\mathrm{d}\, w_a = \int_{\frac{1}{3}}^{1} w_a \,\mathrm{d}\, w_a = \frac{4}{9} \quad \text{and} \quad \int\limits_{\mathsf{V}^t} \mathrm{d}\, w_a = \int_{\frac{1}{3}}^{1} \mathrm{d}\, w_a = \frac{2}{3}.$$

The voting power of class A according to AWTI equals $\frac{2}{3}$, which is the power of the first voter. The power of class B equals $\frac{1}{3}$. Since all voters comprising a class share its power equally, the AWTI power vector for the voters reads $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

We now turn to the final index. The ARTI polytope is given by

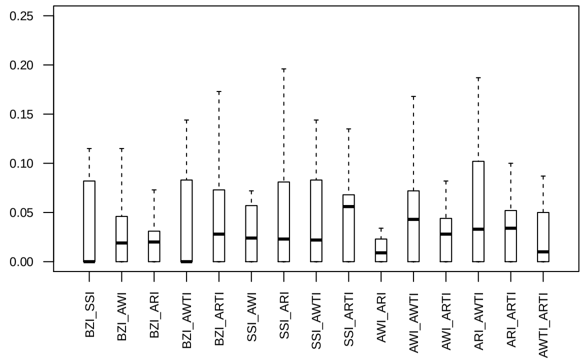
$$\mathsf{R}^t(v) = \left\{(q, w_a) \in \mathbb{R}^2_{\geq 0} \mid 3w_a \geq 1, 2q \leq 1 + w_a, q \geq 1 - w_a, q \geq w_a\right\},$$

where w_a is the voting weight of class A. We have,

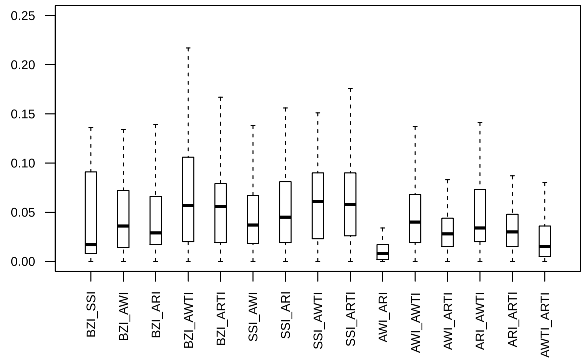
$$\begin{aligned}
\iint\limits_{\mathsf{R}^t} w_a \,\mathrm{d}\, w_a \,\mathrm{d}\, q &= \int_{\frac{1}{3}}^{\frac{1}{2}} w_a \,\mathrm{d}\, w_a \int_{1-w_a}^{\frac{1+w_a}{2}} \mathrm{d}\, q + \int_{\frac{1}{2}}^{1} w_a \,\mathrm{d}\, w_a \int_{w_a}^{\frac{1+w_a}{2}} \mathrm{d}\, q = \frac{1}{108} + \frac{1}{24} = \frac{11}{216}, \\
\iint\limits_{\mathsf{R}^t} \mathrm{d}\, w_a \,\mathrm{d}\, q &= \int_{\frac{1}{3}}^{\frac{1}{2}} \mathrm{d}\, w_a \int_{1-w_a}^{\frac{1+w_a}{2}} \mathrm{d}\, q + \int_{\frac{1}{2}}^{1} \mathrm{d}\, w_a \int_{w_a}^{\frac{1+w_a}{2}} \mathrm{d}\, q = \frac{1}{48} + \frac{1}{16} = \frac{1}{12}.
\end{aligned}$$

The power distribution according to the ARTI is $\left(\frac{11}{18}, \frac{7}{36}, \frac{7}{36}\right)$.

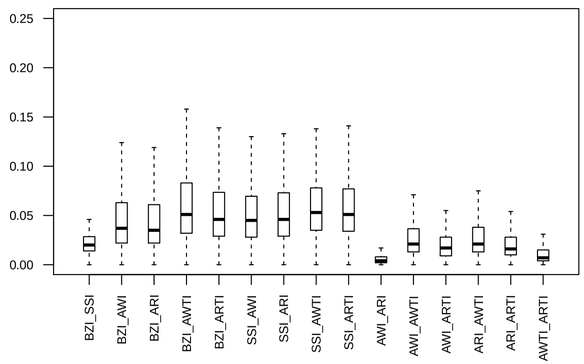
A.2 The distribution of Euclidean distances between the indices



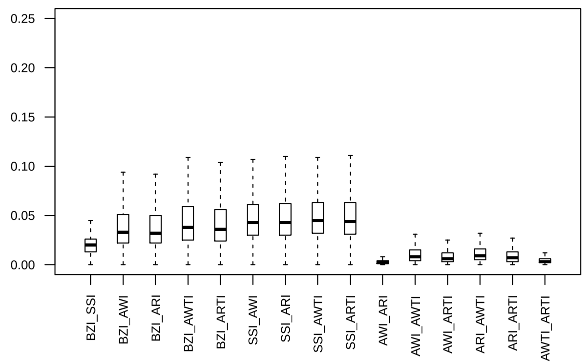
(a) $n \leq 4$



(b) $n \leq 5$



(c) $n \leq 6$



(d) $n \leq 7$

A.3 Representation-compatible power indices in small games

Table A.1: Representation-compatible power indices for $n \leq 5$.

Game	AWI	ARI	Game	AWI	ARI
[1; 1, 0, 0, 0, 0]	(1.000, 0.000, 0.000, 0.000, 0.000)	(1.000, 0.000, 0.000, 0.000, 0.000)	[5; 3, 1, 1, 1, 1]	(0.502, 0.125, 0.125, 0.125, 0.125)	(0.489, 0.128, 0.128, 0.128, 0.128)
[1; 1, 1, 0, 0, 0]	(0.500, 0.500, 0.000, 0.000, 0.000)	(0.500, 0.500, 0.000, 0.000, 0.000)	[5; 3, 2, 2, 2, 1]	(0.300, 0.198, 0.198, 0.198, 0.104)	(0.300, 0.199, 0.199, 0.199, 0.103)
[1; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[5; 4, 1, 1, 1, 1]	(0.586, 0.104, 0.104, 0.104, 0.104)	(0.571, 0.107, 0.107, 0.107, 0.107)
[1; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[5; 4, 2, 2, 1, 1]	(0.424, 0.209, 0.209, 0.079, 0.079)	(0.420, 0.207, 0.207, 0.083, 0.083)
[1; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[5; 4, 3, 2, 1, 1]	(0.382, 0.297, 0.173, 0.074, 0.074)	(0.379, 0.293, 0.174, 0.077, 0.077)
[2; 1, 1, 0, 0, 0]	(0.500, 0.500, 0.000, 0.000, 0.000)	(0.500, 0.500, 0.000, 0.000, 0.000)	[5; 4, 3, 2, 2, 1]	(0.324, 0.258, 0.169, 0.169, 0.079)	(0.326, 0.256, 0.169, 0.169, 0.080)
[2; 2, 1, 1, 0, 0]	(0.611, 0.194, 0.194, 0.000, 0.000)	(0.583, 0.208, 0.208, 0.000, 0.000)	[5; 5, 2, 2, 1, 1]	(0.555, 0.172, 0.172, 0.050, 0.050)	(0.538, 0.174, 0.174, 0.057, 0.057)
[2; 2, 1, 1, 1, 0]	(0.479, 0.174, 0.174, 0.174, 0.000)	(0.463, 0.179, 0.179, 0.179, 0.000)	[5; 5, 3, 2, 1, 1]	(0.518, 0.247, 0.138, 0.048, 0.048)	(0.501, 0.247, 0.143, 0.054, 0.054)
[2; 2, 1, 1, 1, 1]	(0.397, 0.151, 0.151, 0.151, 0.151)	(0.387, 0.153, 0.153, 0.153, 0.153)	[5; 5, 3, 2, 2, 1]	(0.478, 0.211, 0.134, 0.134, 0.043)	(0.463, 0.214, 0.138, 0.138, 0.049)
[2; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[5; 2, 2, 2, 1, 1]	(0.256, 0.256, 0.256, 0.116, 0.116)	(0.255, 0.255, 0.255, 0.117, 0.117)
[2; 2, 2, 1, 1, 0]	(0.396, 0.396, 0.104, 0.104, 0.000)	(0.383, 0.383, 0.117, 0.117, 0.000)	[5; 3, 3, 2, 1, 1]	(0.319, 0.319, 0.200, 0.081, 0.081)	(0.316, 0.316, 0.200, 0.084, 0.084)
[2; 2, 2, 1, 1, 1]	(0.340, 0.340, 0.107, 0.107, 0.107)	(0.331, 0.331, 0.113, 0.113, 0.113)	[5; 3, 3, 2, 2, 1]	(0.286, 0.286, 0.189, 0.189, 0.050)	(0.284, 0.284, 0.188, 0.188, 0.057)
[2; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[6; 2, 2, 2, 1, 1]	(0.249, 0.249, 0.249, 0.127, 0.127)	(0.249, 0.249, 0.249, 0.127, 0.127)
[2; 2, 2, 2, 1, 1]	(0.290, 0.290, 0.290, 0.065, 0.065)	(0.283, 0.283, 0.283, 0.075, 0.075)	[6; 2, 2, 1, 1, 1]	(0.340, 0.340, 0.107, 0.107, 0.107)	(0.331, 0.331, 0.113, 0.113, 0.113)
[2; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[6; 3, 2, 1, 1, 1]	(0.457, 0.237, 0.102, 0.102, 0.102)	(0.443, 0.239, 0.106, 0.106, 0.106)
[3; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[6; 4, 2, 1, 1, 1]	(0.532, 0.214, 0.085, 0.085, 0.085)	(0.517, 0.215, 0.089, 0.089, 0.089)
[3; 2, 1, 1, 0, 0]	(0.611, 0.194, 0.194, 0.000, 0.000)	(0.583, 0.208, 0.208, 0.000, 0.000)	[6; 3, 3, 1, 1, 1]	(0.353, 0.353, 0.098, 0.098, 0.098)	(0.350, 0.350, 0.100, 0.100, 0.100)
[3; 2, 1, 1, 1, 0]	(0.438, 0.188, 0.188, 0.188, 0.000)	(0.430, 0.190, 0.190, 0.190, 0.000)	[6; 3, 3, 2, 1, 1]	(0.319, 0.319, 0.200, 0.081, 0.081)	(0.316, 0.316, 0.200, 0.084, 0.084)
[3; 2, 1, 1, 1, 1]	(0.345, 0.164, 0.164, 0.164, 0.164)	(0.343, 0.164, 0.164, 0.164, 0.164)	[6; 3, 3, 2, 2, 2]	(0.247, 0.247, 0.169, 0.169, 0.169)	(0.248, 0.248, 0.168, 0.168, 0.168)
[3; 3, 1, 1, 1, 0]	(0.600, 0.133, 0.133, 0.133, 0.000)	(0.580, 0.140, 0.140, 0.140, 0.000)	[6; 3, 2, 2, 1, 1]	(0.333, 0.221, 0.221, 0.112, 0.112)	(0.333, 0.221, 0.221, 0.112, 0.112)
[3; 3, 2, 1, 1, 0]	(0.535, 0.270, 0.098, 0.098, 0.000)	(0.513, 0.273, 0.107, 0.107, 0.000)	[6; 4, 2, 2, 1, 1]	(0.424, 0.209, 0.209, 0.079, 0.079)	(0.420, 0.207, 0.207, 0.083, 0.083)
[3; 3, 2, 1, 1, 1]	(0.457, 0.237, 0.102, 0.102, 0.102)	(0.443, 0.239, 0.106, 0.106, 0.106)	[6; 3, 2, 2, 2, 1]	(0.300, 0.198, 0.198, 0.198, 0.104)	(0.300, 0.199, 0.199, 0.199, 0.103)
[3; 3, 1, 1, 1, 1]	(0.502, 0.125, 0.125, 0.125, 0.125)	(0.489, 0.128, 0.128, 0.128, 0.128)	[6; 4, 3, 3, 1, 1]	(0.367, 0.261, 0.261, 0.056, 0.056)	(0.361, 0.259, 0.259, 0.060, 0.060)
[3; 3, 2, 2, 1, 1]	(0.424, 0.198, 0.198, 0.090, 0.090)	(0.409, 0.202, 0.202, 0.093, 0.093)	[6; 4, 3, 3, 2, 1]	(0.299, 0.238, 0.238, 0.150, 0.075)	(0.300, 0.237, 0.237, 0.151, 0.076)
[3; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[6; 4, 3, 2, 2, 1]	(0.354, 0.275, 0.152, 0.152, 0.067)	(0.350, 0.271, 0.154, 0.154, 0.070)
[3; 2, 2, 1, 1, 0]	(0.346, 0.346, 0.154, 0.154, 0.000)	(0.343, 0.343, 0.157, 0.157, 0.000)	[6; 5, 2, 2, 2, 1]	(0.449, 0.169, 0.169, 0.169, 0.045)	(0.444, 0.168, 0.168, 0.168, 0.052)
[3; 2, 2, 1, 1, 1]	(0.295, 0.295, 0.136, 0.136, 0.136)	(0.294, 0.294, 0.138, 0.138, 0.138)	[6; 5, 3, 3, 1, 1]	(0.416, 0.245, 0.245, 0.047, 0.047)	(0.411, 0.243, 0.243, 0.052, 0.052)
[3; 3, 3, 1, 1, 1]	(0.390, 0.390, 0.073, 0.073, 0.073)	(0.381, 0.381, 0.079, 0.079, 0.079)	[6; 5, 4, 2, 2, 1]	(0.374, 0.323, 0.132, 0.132, 0.039)	(0.371, 0.317, 0.134, 0.134, 0.045)
[3; 3, 3, 2, 1, 1]	(0.364, 0.364, 0.155, 0.059, 0.059)	(0.353, 0.353, 0.163, 0.065, 0.065)	[7; 2, 2, 2, 1, 1]	(0.290, 0.290, 0.290, 0.065, 0.065)	(0.283, 0.283, 0.283, 0.075, 0.075)
[3; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[7; 3, 2, 2, 1, 1]	(0.424, 0.198, 0.198, 0.090, 0.090)	(0.409, 0.202, 0.202, 0.093, 0.093)

Table A.2: Representation-compatible power indices for $n \leq 5$ (cont.).

Game	AWI	ARI	Game	AWI	ARI
[3; 2, 2, 2, 1, 1]	(0.249, 0.249, 0.249, 0.127, 0.127)	(0.249, 0.249, 0.249, 0.127, 0.127)	[7; 3, 3, 1, 1, 1]	(0.390, 0.390, 0.073, 0.073, 0.073)	(0.381, 0.381, 0.079, 0.079, 0.079)
[4; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[7; 3, 3, 2, 2, 1]	(0.286, 0.286, 0.189, 0.189, 0.050)	(0.284, 0.284, 0.188, 0.188, 0.057)
[4; 2, 2, 1, 1, 0]	(0.346, 0.346, 0.154, 0.154, 0.000)	(0.343, 0.343, 0.157, 0.157, 0.000)	[7; 4, 3, 1, 1, 1]	(0.495, 0.290, 0.072, 0.072, 0.072)	(0.479, 0.292, 0.076, 0.076, 0.076)
[4; 2, 2, 1, 1, 1]	(0.300, 0.300, 0.133, 0.133, 0.133)	(0.298, 0.298, 0.135, 0.135, 0.135)	[7; 4, 3, 2, 1, 1]	(0.382, 0.297, 0.173, 0.074, 0.074)	(0.379, 0.293, 0.174, 0.077, 0.077)
[4; 2, 1, 1, 1, 0]	(0.479, 0.174, 0.174, 0.174, 0.000)	(0.463, 0.179, 0.179, 0.179, 0.000)	[7; 4, 3, 2, 2, 1]	(0.354, 0.275, 0.152, 0.152, 0.067)	(0.350, 0.271, 0.154, 0.154, 0.070)
[4; 2, 1, 1, 1, 1]	(0.345, 0.164, 0.164, 0.164, 0.164)	(0.343, 0.164, 0.164, 0.164, 0.164)	[7; 3, 2, 2, 2, 1]	(0.310, 0.212, 0.212, 0.212, 0.053)	(0.308, 0.210, 0.210, 0.210, 0.061)
[4; 3, 1, 1, 1, 0]	(0.600, 0.133, 0.133, 0.133, 0.000)	(0.580, 0.140, 0.140, 0.140, 0.000)	[7; 4, 2, 2, 1, 1]	(0.488, 0.177, 0.177, 0.078, 0.078)	(0.474, 0.181, 0.181, 0.082, 0.082)
[4; 3, 2, 2, 1, 0]	(0.402, 0.258, 0.258, 0.081, 0.000)	(0.397, 0.257, 0.257, 0.090, 0.000)	[7; 5, 2, 2, 1, 1]	(0.555, 0.172, 0.172, 0.050, 0.050)	(0.538, 0.174, 0.174, 0.057, 0.057)
[4; 3, 1, 1, 1, 1]	(0.467, 0.133, 0.133, 0.133, 0.133)	(0.460, 0.135, 0.135, 0.135, 0.135)	[7; 4, 3, 3, 1, 1]	(0.367, 0.261, 0.261, 0.056, 0.056)	(0.361, 0.259, 0.259, 0.060, 0.060)
[4; 3, 2, 2, 1, 1]	(0.333, 0.221, 0.221, 0.112, 0.112)	(0.333, 0.221, 0.221, 0.112, 0.112)	[7; 4, 3, 3, 2, 2]	(0.273, 0.218, 0.218, 0.146, 0.146)	(0.275, 0.217, 0.217, 0.145, 0.145)
[4; 3, 2, 1, 1, 1]	(0.391, 0.259, 0.117, 0.117, 0.117)	(0.388, 0.257, 0.118, 0.118, 0.118)	[7; 5, 2, 2, 2, 1]	(0.449, 0.169, 0.169, 0.169, 0.045)	(0.444, 0.168, 0.168, 0.168, 0.052)
[4; 3, 2, 2, 2, 1]	(0.310, 0.212, 0.212, 0.212, 0.053)	(0.308, 0.210, 0.210, 0.210, 0.061)	[7; 5, 3, 3, 2, 1]	(0.392, 0.225, 0.225, 0.117, 0.041)	(0.386, 0.223, 0.223, 0.121, 0.046)
[4; 4, 1, 1, 1, 1]	(0.586, 0.104, 0.104, 0.104, 0.104)	(0.571, 0.107, 0.107, 0.107, 0.107)	[7; 5, 4, 3, 2, 1]	(0.354, 0.305, 0.190, 0.114, 0.037)	(0.350, 0.299, 0.191, 0.117, 0.042)
[4; 4, 2, 2, 1, 1]	(0.488, 0.177, 0.177, 0.078, 0.078)	(0.474, 0.181, 0.181, 0.082, 0.082)	[7; 3, 3, 2, 2, 2]	(0.247, 0.247, 0.169, 0.169, 0.169)	(0.248, 0.248, 0.168, 0.168, 0.168)
[4; 4, 2, 1, 1, 1]	(0.532, 0.214, 0.085, 0.085, 0.085)	(0.517, 0.215, 0.089, 0.089, 0.089)	[8; 3, 3, 2, 1, 1]	(0.364, 0.364, 0.155, 0.059, 0.059)	(0.353, 0.353, 0.163, 0.065, 0.065)
[4; 4, 3, 1, 1, 1]	(0.495, 0.290, 0.072, 0.072, 0.072)	(0.479, 0.292, 0.076, 0.076, 0.076)	[8; 3, 3, 2, 2, 1]	(0.276, 0.276, 0.182, 0.182, 0.084)	(0.275, 0.275, 0.182, 0.182, 0.085)
[4; 4, 3, 2, 2, 1]	(0.440, 0.224, 0.145, 0.145, 0.045)	(0.423, 0.228, 0.149, 0.149, 0.052)	[8; 4, 3, 2, 2, 1]	(0.324, 0.258, 0.169, 0.169, 0.079)	(0.326, 0.256, 0.169, 0.169, 0.080)
[4; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[8; 5, 3, 2, 1, 1]	(0.518, 0.247, 0.138, 0.048, 0.048)	(0.501, 0.247, 0.143, 0.054, 0.054)
[4; 2, 2, 2, 1, 1]	(0.256, 0.256, 0.256, 0.116, 0.116)	(0.255, 0.255, 0.255, 0.117, 0.117)	[8; 4, 3, 3, 2, 1]	(0.299, 0.238, 0.238, 0.150, 0.075)	(0.300, 0.237, 0.237, 0.151, 0.076)
[4; 3, 3, 1, 1, 1]	(0.353, 0.353, 0.098, 0.098, 0.098)	(0.350, 0.350, 0.100, 0.100, 0.100)	[8; 5, 3, 3, 2, 1]	(0.392, 0.225, 0.225, 0.117, 0.041)	(0.386, 0.223, 0.223, 0.121, 0.046)
[4; 3, 3, 2, 2, 1]	(0.276, 0.276, 0.182, 0.182, 0.084)	(0.275, 0.275, 0.182, 0.182, 0.085)	[8; 5, 3, 3, 1, 1]	(0.416, 0.245, 0.245, 0.047, 0.047)	(0.411, 0.243, 0.243, 0.052, 0.052)
[5; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[8; 5, 4, 3, 2, 2]	(0.328, 0.285, 0.169, 0.109, 0.109)	(0.325, 0.279, 0.172, 0.112, 0.112)
[5; 2, 2, 1, 1, 0]	(0.396, 0.396, 0.104, 0.104, 0.000)	(0.383, 0.383, 0.117, 0.117, 0.000)	[8; 4, 3, 3, 2, 2]	(0.273, 0.218, 0.218, 0.146, 0.146)	(0.275, 0.217, 0.217, 0.145, 0.145)
[5; 2, 2, 1, 1, 1]	(0.295, 0.295, 0.136, 0.136, 0.136)	(0.294, 0.294, 0.138, 0.138, 0.138)	[9; 4, 3, 2, 2, 1]	(0.440, 0.224, 0.145, 0.145, 0.045)	(0.423, 0.228, 0.149, 0.149, 0.052)
[5; 3, 2, 1, 1, 0]	(0.535, 0.270, 0.098, 0.098, 0.000)	(0.513, 0.273, 0.107, 0.107, 0.000)	[9; 5, 3, 2, 2, 1]	(0.478, 0.211, 0.134, 0.134, 0.043)	(0.463, 0.214, 0.138, 0.138, 0.049)
[5; 3, 2, 1, 1, 1]	(0.391, 0.259, 0.117, 0.117, 0.117)	(0.388, 0.257, 0.118, 0.118, 0.118)	[9; 5, 4, 2, 2, 1]	(0.374, 0.323, 0.132, 0.132, 0.039)	(0.371, 0.317, 0.134, 0.134, 0.045)
[5; 2, 1, 1, 1, 1]	(0.397, 0.151, 0.151, 0.151, 0.151)	(0.387, 0.153, 0.153, 0.153, 0.153)	[9; 5, 4, 3, 2, 1]	(0.354, 0.305, 0.190, 0.114, 0.037)	(0.350, 0.299, 0.191, 0.117, 0.042)
[5; 3, 2, 2, 1, 1]	(0.367, 0.233, 0.233, 0.083, 0.083)	(0.361, 0.231, 0.231, 0.088, 0.088)	[9; 5, 4, 3, 2, 2]	(0.328, 0.285, 0.169, 0.109, 0.109)	(0.325, 0.279, 0.172, 0.112, 0.112)
[5; 3, 2, 2, 1, 0]	(0.402, 0.258, 0.258, 0.081, 0.000)	(0.397, 0.257, 0.257, 0.090, 0.000)			

Table A.3: Representation-compatible power indices for $n \leq 5$.

Game	AWTI	ARTI	Game	AWTI	ARTI
[1; 1, 0, 0, 0, 0]	(1.000, 0.000, 0.000, 0.000, 0.000)	(1.000, 0.000, 0.000, 0.000, 0.000)	[5; 3, 1, 1, 1, 1]	(0.667, 0.083, 0.083, 0.083, 0.083)	(0.587, 0.103, 0.103, 0.103, 0.103)
[1; 1, 1, 0, 0, 0]	(0.500, 0.500, 0.000, 0.000, 0.000)	(0.500, 0.500, 0.000, 0.000, 0.000)	[5; 3, 2, 2, 2, 1]	(0.294, 0.206, 0.206, 0.206, 0.089)	(0.296, 0.204, 0.204, 0.204, 0.092)
[1; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[5; 4, 1, 1, 1, 1]	(0.714, 0.071, 0.071, 0.071, 0.071)	(0.643, 0.089, 0.089, 0.089, 0.089)
[1; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[5; 4, 2, 2, 1, 1]	(0.407, 0.198, 0.198, 0.099, 0.099)	(0.405, 0.199, 0.199, 0.099, 0.099)
[1; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[5; 4, 3, 2, 1, 1]	(0.378, 0.321, 0.166, 0.068, 0.068)	(0.375, 0.311, 0.169, 0.072, 0.072)
[2; 1, 1, 0, 0, 0]	(0.500, 0.500, 0.000, 0.000, 0.000)	(0.500, 0.500, 0.000, 0.000, 0.000)	[5; 4, 3, 2, 2, 1]	(0.322, 0.239, 0.170, 0.170, 0.099)	(0.324, 0.241, 0.169, 0.169, 0.096)
[2; 2, 1, 1, 0, 0]	(0.667, 0.167, 0.167, 0.000, 0.000)	(0.611, 0.194, 0.194, 0.000, 0.000)	[5; 5, 2, 2, 1, 1]	(0.625, 0.137, 0.137, 0.051, 0.051)	(0.582, 0.148, 0.148, 0.061, 0.061)
[2; 2, 1, 1, 1, 0]	(0.625, 0.125, 0.125, 0.125, 0.000)	(0.550, 0.150, 0.150, 0.150, 0.000)	[5; 5, 3, 2, 1, 1]	(0.548, 0.258, 0.123, 0.035, 0.035)	(0.522, 0.257, 0.132, 0.045, 0.045)
[2; 2, 1, 1, 1, 1]	(0.600, 0.100, 0.100, 0.100, 0.100)	(0.511, 0.122, 0.122, 0.122, 0.122)	[5; 5, 3, 2, 2, 1]	(0.514, 0.180, 0.126, 0.126, 0.054)	(0.488, 0.190, 0.132, 0.132, 0.058)
[2; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[5; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)
[2; 2, 2, 1, 1, 0]	(0.375, 0.375, 0.125, 0.125, 0.000)	(0.361, 0.361, 0.139, 0.139, 0.000)	[5; 3, 3, 2, 1, 1]	(0.356, 0.356, 0.175, 0.056, 0.056)	(0.342, 0.342, 0.182, 0.067, 0.067)
[2; 2, 2, 1, 1, 1]	(0.350, 0.350, 0.100, 0.100, 0.100)	(0.329, 0.329, 0.114, 0.114, 0.114)	[5; 3, 3, 2, 2, 1]	(0.265, 0.265, 0.193, 0.193, 0.084)	(0.267, 0.267, 0.190, 0.190, 0.086)
[2; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[6; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)
[2; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)	[6; 2, 2, 1, 1, 1]	(0.350, 0.350, 0.100, 0.100, 0.100)	(0.329, 0.329, 0.114, 0.114, 0.114)
[2; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[6; 3, 2, 1, 1, 1]	(0.540, 0.173, 0.096, 0.096, 0.096)	(0.499, 0.192, 0.103, 0.103, 0.103)
[3; 1, 1, 1, 0, 0]	(0.333, 0.333, 0.333, 0.000, 0.000)	(0.333, 0.333, 0.333, 0.000, 0.000)	[6; 4, 2, 1, 1, 1]	(0.620, 0.224, 0.052, 0.052, 0.052)	(0.576, 0.223, 0.067, 0.067, 0.067)
[3; 2, 1, 1, 0, 0]	(0.667, 0.167, 0.167, 0.000, 0.000)	(0.611, 0.194, 0.194, 0.000, 0.000)	[6; 3, 3, 1, 1, 1]	(0.393, 0.393, 0.071, 0.071, 0.071)	(0.373, 0.373, 0.085, 0.085, 0.085)
[3; 2, 1, 1, 1, 0]	(0.375, 0.208, 0.208, 0.208, 0.000)	(0.383, 0.206, 0.206, 0.206, 0.000)	[6; 3, 3, 2, 1, 1]	(0.356, 0.356, 0.175, 0.056, 0.056)	(0.342, 0.342, 0.182, 0.067, 0.067)
[3; 2, 1, 1, 1, 1]	(0.314, 0.171, 0.171, 0.171, 0.171)	(0.321, 0.170, 0.170, 0.170, 0.170)	[6; 3, 3, 2, 2, 2]	(0.243, 0.243, 0.171, 0.171, 0.171)	(0.245, 0.245, 0.170, 0.170, 0.170)
[3; 3, 1, 1, 1, 0]	(0.700, 0.100, 0.100, 0.100, 0.000)	(0.633, 0.122, 0.122, 0.122, 0.000)	[6; 3, 2, 2, 1, 1]	(0.311, 0.233, 0.233, 0.111, 0.111)	(0.317, 0.231, 0.231, 0.111, 0.111)
[3; 3, 2, 1, 1, 0]	(0.602, 0.249, 0.075, 0.075, 0.000)	(0.558, 0.258, 0.092, 0.092, 0.000)	[6; 4, 2, 2, 1, 1]	(0.407, 0.198, 0.198, 0.099, 0.099)	(0.405, 0.199, 0.199, 0.099, 0.099)
[3; 3, 2, 1, 1, 1]	(0.540, 0.173, 0.096, 0.096, 0.096)	(0.499, 0.192, 0.103, 0.103, 0.103)	[6; 3, 2, 2, 2, 1]	(0.294, 0.206, 0.206, 0.206, 0.089)	(0.296, 0.204, 0.204, 0.204, 0.092)
[3; 3, 1, 1, 1, 1]	(0.667, 0.083, 0.083, 0.083, 0.083)	(0.587, 0.103, 0.103, 0.103, 0.103)	[6; 4, 3, 3, 1, 1]	(0.361, 0.269, 0.269, 0.051, 0.051)	(0.354, 0.264, 0.264, 0.059, 0.059)
[3; 3, 2, 2, 1, 1]	(0.528, 0.176, 0.176, 0.060, 0.060)	(0.479, 0.188, 0.188, 0.073, 0.073)	[6; 4, 3, 3, 2, 1]	(0.302, 0.244, 0.244, 0.152, 0.059)	(0.303, 0.241, 0.241, 0.152, 0.062)
[3; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[6; 4, 3, 2, 2, 1]	(0.359, 0.288, 0.148, 0.148, 0.057)	(0.354, 0.280, 0.152, 0.152, 0.062)
[3; 2, 2, 1, 1, 0]	(0.375, 0.375, 0.125, 0.125, 0.000)	(0.361, 0.361, 0.139, 0.139, 0.000)	[6; 5, 2, 2, 2, 1]	(0.415, 0.170, 0.170, 0.170, 0.075)	(0.415, 0.169, 0.169, 0.169, 0.077)
[3; 2, 2, 1, 1, 1]	(0.267, 0.267, 0.156, 0.156, 0.156)	(0.273, 0.273, 0.151, 0.151, 0.151)	[6; 5, 3, 3, 1, 1]	(0.394, 0.258, 0.258, 0.045, 0.045)	(0.392, 0.251, 0.251, 0.053, 0.053)
[3; 3, 3, 1, 1, 1]	(0.393, 0.393, 0.071, 0.071, 0.071)	(0.373, 0.373, 0.085, 0.085, 0.085)	[6; 5, 4, 2, 2, 1]	(0.384, 0.321, 0.123, 0.123, 0.049)	(0.379, 0.314, 0.127, 0.127, 0.054)
[3; 3, 3, 2, 1, 1]	(0.356, 0.356, 0.175, 0.056, 0.056)	(0.342, 0.342, 0.182, 0.067, 0.067)	[7; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)
[3; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[7; 3, 2, 2, 1, 1]	(0.528, 0.176, 0.176, 0.060, 0.060)	(0.479, 0.188, 0.188, 0.073, 0.073)

Table A.4: Representation-compatible power indices for $n \leq 5$ (cont.).

Game	AWTI	ARTI	Game	AWTI	ARTI
[3; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)	[7; 3, 3, 1, 1, 1]	(0.393, 0.393, 0.071, 0.071, 0.071)	(0.373, 0.373, 0.085, 0.085, 0.085)
[4; 1, 1, 1, 1, 0]	(0.250, 0.250, 0.250, 0.250, 0.000)	(0.250, 0.250, 0.250, 0.250, 0.000)	[7; 3, 3, 2, 2, 1]	(0.265, 0.265, 0.193, 0.193, 0.084)	(0.267, 0.267, 0.190, 0.190, 0.086)
[4; 2, 2, 1, 1, 0]	(0.375, 0.375, 0.125, 0.125, 0.000)	(0.361, 0.361, 0.139, 0.139, 0.000)	[7; 4, 3, 1, 1, 1]	(0.603, 0.261, 0.045, 0.045, 0.045)	(0.552, 0.271, 0.059, 0.059, 0.059)
[4; 2, 2, 1, 1, 1]	(0.375, 0.375, 0.083, 0.083, 0.083)	(0.345, 0.345, 0.103, 0.103, 0.103)	[7; 4, 3, 2, 1, 1]	(0.378, 0.321, 0.166, 0.068, 0.068)	(0.375, 0.311, 0.169, 0.072, 0.072)
[4; 2, 1, 1, 1, 0]	(0.625, 0.125, 0.125, 0.125, 0.000)	(0.550, 0.150, 0.150, 0.150, 0.000)	[7; 4, 3, 2, 2, 1]	(0.359, 0.288, 0.148, 0.148, 0.057)	(0.354, 0.280, 0.152, 0.152, 0.062)
[4; 2, 1, 1, 1, 1]	(0.314, 0.171, 0.171, 0.171, 0.171)	(0.321, 0.170, 0.170, 0.170, 0.170)	[7; 3, 2, 2, 2, 1]	(0.294, 0.206, 0.206, 0.206, 0.089)	(0.296, 0.204, 0.204, 0.204, 0.092)
[4; 3, 1, 1, 1, 0]	(0.700, 0.100, 0.100, 0.100, 0.000)	(0.633, 0.122, 0.122, 0.122, 0.000)	[7; 4, 2, 2, 1, 1]	(0.556, 0.167, 0.167, 0.056, 0.056)	(0.517, 0.175, 0.175, 0.067, 0.067)
[4; 3, 2, 2, 1, 0]	(0.369, 0.261, 0.261, 0.108, 0.000)	(0.371, 0.258, 0.258, 0.113, 0.000)	[7; 5, 2, 2, 1, 1]	(0.625, 0.137, 0.137, 0.051, 0.051)	(0.582, 0.148, 0.148, 0.061, 0.061)
[4; 3, 1, 1, 1, 1]	(0.417, 0.146, 0.146, 0.146, 0.146)	(0.421, 0.145, 0.145, 0.145, 0.145)	[7; 4, 3, 3, 1, 1]	(0.361, 0.269, 0.269, 0.051, 0.051)	(0.354, 0.264, 0.264, 0.059, 0.059)
[4; 3, 2, 2, 1, 1]	(0.311, 0.233, 0.233, 0.111, 0.111)	(0.317, 0.231, 0.231, 0.111, 0.111)	[7; 4, 3, 3, 2, 2]	(0.277, 0.208, 0.208, 0.153, 0.153)	(0.279, 0.210, 0.210, 0.151, 0.151)
[4; 3, 2, 1, 1, 1]	(0.397, 0.294, 0.103, 0.103, 0.103)	(0.391, 0.283, 0.109, 0.109, 0.109)	[7; 5, 2, 2, 2, 1]	(0.415, 0.170, 0.170, 0.170, 0.075)	(0.415, 0.169, 0.169, 0.169, 0.077)
[4; 3, 2, 2, 2, 1]	(0.294, 0.206, 0.206, 0.206, 0.089)	(0.296, 0.204, 0.204, 0.204, 0.092)	[7; 5, 3, 3, 2, 1]	(0.365, 0.219, 0.219, 0.146, 0.052)	(0.363, 0.218, 0.218, 0.146, 0.056)
[4; 4, 1, 1, 1, 1]	(0.714, 0.071, 0.071, 0.071, 0.071)	(0.643, 0.089, 0.089, 0.089, 0.089)	[7; 5, 4, 3, 2, 1]	(0.354, 0.305, 0.190, 0.114, 0.037)	(0.350, 0.299, 0.191, 0.117, 0.042)
[4; 4, 2, 2, 1, 1]	(0.556, 0.167, 0.167, 0.056, 0.056)	(0.517, 0.175, 0.175, 0.067, 0.067)	[7; 3, 3, 2, 2, 2]	(0.243, 0.243, 0.171, 0.171, 0.171)	(0.245, 0.245, 0.170, 0.170, 0.170)
[4; 4, 2, 1, 1, 1]	(0.620, 0.224, 0.052, 0.052, 0.052)	(0.576, 0.223, 0.067, 0.067, 0.067)	[8; 3, 3, 2, 1, 1]	(0.356, 0.356, 0.175, 0.056, 0.056)	(0.342, 0.342, 0.182, 0.067, 0.067)
[4; 4, 3, 1, 1, 1]	(0.603, 0.261, 0.045, 0.045, 0.045)	(0.552, 0.271, 0.059, 0.059, 0.059)	[8; 3, 3, 2, 2, 1]	(0.265, 0.265, 0.193, 0.193, 0.084)	(0.267, 0.267, 0.190, 0.190, 0.086)
[4; 4, 3, 2, 2, 1]	(0.467, 0.196, 0.140, 0.140, 0.057)	(0.440, 0.207, 0.145, 0.145, 0.062)	[8; 4, 3, 2, 2, 1]	(0.322, 0.239, 0.170, 0.170, 0.099)	(0.324, 0.241, 0.169, 0.169, 0.096)
[4; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[8; 5, 3, 2, 1, 1]	(0.548, 0.258, 0.123, 0.035, 0.035)	(0.522, 0.257, 0.132, 0.045, 0.045)
[4; 2, 2, 2, 1, 1]	(0.267, 0.267, 0.267, 0.100, 0.100)	(0.261, 0.261, 0.261, 0.108, 0.108)	[8; 4, 3, 3, 2, 1]	(0.302, 0.244, 0.244, 0.152, 0.059)	(0.303, 0.241, 0.241, 0.152, 0.062)
[4; 3, 3, 1, 1, 1]	(0.393, 0.393, 0.071, 0.071, 0.071)	(0.373, 0.373, 0.085, 0.085, 0.085)	[8; 5, 3, 3, 2, 1]	(0.365, 0.219, 0.219, 0.146, 0.052)	(0.363, 0.218, 0.218, 0.146, 0.056)
[4; 3, 3, 2, 2, 1]	(0.265, 0.265, 0.193, 0.193, 0.084)	(0.267, 0.267, 0.190, 0.190, 0.086)	[8; 5, 3, 3, 1, 1]	(0.394, 0.258, 0.258, 0.045, 0.045)	(0.392, 0.251, 0.251, 0.053, 0.053)
[5; 1, 1, 1, 1, 1]	(0.200, 0.200, 0.200, 0.200, 0.200)	(0.200, 0.200, 0.200, 0.200, 0.200)	[8; 5, 4, 3, 2, 2]	(0.347, 0.294, 0.149, 0.105, 0.105)	(0.340, 0.285, 0.156, 0.109, 0.109)
[5; 2, 2, 1, 1, 0]	(0.375, 0.375, 0.125, 0.125, 0.000)	(0.361, 0.361, 0.139, 0.139, 0.000)	[8; 4, 3, 3, 2, 2]	(0.277, 0.208, 0.208, 0.153, 0.153)	(0.279, 0.210, 0.210, 0.151, 0.151)
[5; 2, 2, 1, 1, 1]	(0.267, 0.267, 0.156, 0.156, 0.156)	(0.273, 0.273, 0.151, 0.151, 0.151)	[9; 4, 3, 2, 2, 1]	(0.467, 0.196, 0.140, 0.140, 0.057)	(0.440, 0.207, 0.145, 0.145, 0.062)
[5; 3, 2, 1, 1, 0]	(0.602, 0.249, 0.075, 0.075, 0.000)	(0.558, 0.258, 0.092, 0.092, 0.000)	[9; 5, 3, 2, 2, 1]	(0.514, 0.180, 0.126, 0.126, 0.054)	(0.488, 0.190, 0.132, 0.132, 0.058)
[5; 3, 2, 1, 1, 1]	(0.397, 0.294, 0.103, 0.103, 0.103)	(0.391, 0.283, 0.109, 0.109, 0.109)	[9; 5, 4, 2, 2, 1]	(0.384, 0.321, 0.123, 0.123, 0.049)	(0.379, 0.314, 0.127, 0.127, 0.054)
[5; 2, 1, 1, 1, 1]	(0.600, 0.100, 0.100, 0.100, 0.100)	(0.511, 0.122, 0.122, 0.122, 0.122)	[9; 5, 4, 3, 2, 1]	(0.354, 0.305, 0.190, 0.114, 0.037)	(0.350, 0.299, 0.191, 0.117, 0.042)
[5; 3, 2, 2, 1, 1]	(0.361, 0.222, 0.222, 0.097, 0.097)	(0.354, 0.222, 0.222, 0.101, 0.101)	[9; 5, 4, 3, 2, 2]	(0.347, 0.294, 0.149, 0.105, 0.105)	(0.340, 0.285, 0.156, 0.109, 0.109)
[5; 3, 2, 2, 1, 0]	(0.369, 0.261, 0.261, 0.108, 0.000)	(0.371, 0.258, 0.258, 0.113, 0.000)			