Measuring Voting Power in Games with Correlated Votes Using Bahadur's Parametrisation

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Abstract

We introduce a method of measuring voting power in simple voting games with correlated votes using the Bahadur parameterisation. With a method for measuring voting power with correlated votes, we can address a question of practical importance. Given that most of the applied power analysis is carried out with either the Penrose-Banzhaf or the Shapley-Shubik measures of power, what happens when you use these two measures in games with correlated votes? Simulations of all possible voting games with up to six players show that both measures tend to overestimate power when the votes are positively correlated. Yet, in most voting scenarios, the Shapley-Shubik index is closer to the probability of criticality than the Penrose-Banzhaf measure. This also holds for the power distribution in the EU Council of Ministers. Based on these simulations, we conclude that, while the Penrose-Banzhaf measure may be ideal for designing constitutional assemblies, the Shapley-Shubik index is better suited for the analysis of power distributions beyond the constitutional stage.

JEL-Codes: D72
Key Words: simple games, correlated votes, voting power, Council of the European Union

1 Introduction

In his papers, Straffin (1978) suggests a probabilistic interpretation of the widely-used Penrose (1946)-Banzhaf (1965) and Shapley and Shubik (1954) measures of voting power. Straffin's prescription is: "If we believe that voters in a certain body have such common standards, the Shapley-Shubik index might be most appropriate; if we believe voters behave independently, the Banzhaf index is the instrument of choice" (p. 117).

Following common standards implies positive correlation between the votes in favour of a proposal and positive correlation between the votes against the proposal. In this paper, we present Bahadur's (1961) parameterisation as a method of modelling correlation in voting games. This flexible parametrisation admits varying probabilities of affirmative votes and correlation coefficients between them, allowing us to model games with a wide variety of probability distributions.

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Having first established a method for measuring voting power in games with correlations, we ask: How do the two most widely used voting power measures, Penrose-Banzhaf and Shapley-Shubik, compare when used to analyse correlated voting? To make our results as representative as possible, we examine *all* simple voting games with up to a given (small) number of players. Since simple voting games include weighted voting games as a subclass, our results indicate what we can expect in weighted voting games.

The number of simple voting games is so large that we had to restrict our attention to games with up to six players. In the case of six players, there are already 7,828,353 distinct simple voting games. Adding one more player would increase the number of games to 2,414,682,040,997, clearly surpassing our current simulation capabilities.[1] During the course of our study, we carried out over half a trillion power calculations in more than twenty-three billion probabilistic scenarios.

Briefly anticipating our results, we found that both measures tend to overestimate voting power in small games with positively correlated votes, and that the Shapley-Shubik index was considerably closer to the probability of criticality than the Penrose-Banzhaf measure. We also checked to see whether these conclusions were valid for the Council of the European Union under the voting rules stipulated by the Treaty of Nice and the Treaty of Lisbon (2014 onwards). Our calculations for the Council attest to the robustness of the Shapley-Shubik index to positive correlation.

The material presented in this paper is organised as follows. Section 2 recapitulates the theoretical foundations of power indices for simple voting games and summarises Straffin's probabilistic interpretations. Bahadur's parameterisation is discussed in Section 3. Sections 4 and 5 summarise our findings for all simple voting games up to six players and for the distribution of power in the Council under the Treaties of Nice and Lisbon. The final section concludes the paper with a brief discussion of the implications of our findings.

2 Simple Voting Games

A *simple voting game* is a collection $\mathcal{W}$ of subsets of a finite set N, satisfying the following properties:

i). $N \in \mathcal{W}$;

ii). $\emptyset \notin \mathcal{W}$;

iii). (Monotonicity) Whenever $S \subseteq T \subseteq N$ and $S \in \mathcal{W}$, then also $T \in \mathcal{W}$.

We shall refer to N, the largest set of $\mathcal{W}$, as the *assembly* of $\mathcal{W}$. The members of N are the *voters* or *players* in $\mathcal{W}$. A set of voters, a subset of N, is called a *coalition*. The *cardinal* of a set of voters S (or the number of voters in coalition S) is denoted by $|S|$.

In the above definition, $\mathcal{W}$ is the set of all winning coalitions. The set of winning coalitions completely characterises the game. As we can use a characteristic function to uniquely identify a set of winning coalitions, it follows that we can use such a function to identify a game. Let $\mathcal{W}$ be the set of winning coalitions from a simple voting game with assembly N. The characteristic

[1] The number of distinct SVGs equals the number of positively monotonic Boolean functions – the Dedekind number, less one to exclude the empty set being a winning coalition. The Dedekind numbers form a rapidly-growing sequence of integers, with only the first nine terms computed to date.

function of $\mathcal{W}$ is the map $\mathbf{w}$ from the set of all coalitions such that, for any coalition $S \subseteq N$,

$$\mathbf{w}(S) = \begin{cases} 1 & S \in \mathcal{W}; \\ 0 & \text{otherwise.} \end{cases}$$

Since each player i can either be a member of a coalition or not, i's action can be expressed by a binary variable v_i, such that $v_i(S) = 1$ if $i \in S$, and $v_i(S) = 0$ if $i \notin S$. The n-tuple of votes $\mathbf{v}(S) = (v_1(S), \ldots, v_n(S))$ is called a voting profile. When there is no risk of confusion about S, we will use the more economical notation $\mathbf{v} = (v_1, \ldots, v_n)$. For n players there would be 2^n such voting profiles.

Voting power measures the ability of a player to be critical, or pivotal, to the outcome of a vote. In a simple voting game there are only two distinct ways in which a player can be critical. He can be critical if he can make a losing coalition win by adding support, or he can be critical if he can make a winning coalition lose by removing support.

The ability of a voter i to be critical in a voting scenario can be measured by:

$$\sum_{T \subset N\ ,\ i \notin T} \pi_T[\mathbf{w}(T \cup \{i\}) - \mathbf{w}(T)] \quad + \sum_{S \subseteq N\ ,\ i \in S} \pi_S[\mathbf{w}(S) - \mathbf{w}(S \setminus \{i\})], \tag{1}$$

where π_T and π_S are the probability of occurrence of coalitions T and S, such that $T = S \setminus \{i\}$.

The above formula offers a general expression for the probability of casting a decisive vote. This general probabilistic view on voting power is well-established in the literature (e.g., Felsenthal and Machover 1998), owing largely to seminal contribution by Straffin (1977).

2.1 Straffin's probabilistic models

Straffin (1977) proves two well-known characterisation theorems based on two stochastic models of votes formulated in the Independence Assumption and the Homogeneity Assumption. Let p_i be the probability that player i votes Yes. "*Independence Assumption*: The p_i's are selected independently from the uniform distribution on $[0, 1]$." "*Homogeneity Assumption*: A number p is selected from the uniform distribution on $[0, 1]$, and $p_i = p$ for all i" (p. 112).

The **Independence Assumption** assumes that all voting profiles are equally probable and occur with the probability $\pi_{\mathbf{v}} = \frac{1}{2^n}$. Substituting these probabilities in Eq. 1 and setting $T = S \setminus \{i\}$ and $n = |N|$ yields the Penrose-Banzhaf measure:

$$\beta'(i) = \frac{1}{2^{|N|-1}} \sum_{S \subseteq N\ ,\ i \in S} [\mathbf{w}(S) - \mathbf{w}(S \setminus \{i\})]. \tag{2}$$

The relative Banzhaf measure, known as the Banzhaf index, is obtained by normalising absolute powers to sum to unity. The Banzhaf index has no probabilistic interpretation and it will not be discussed further here. Normalisation is appropriate when power justifies a claim on a prize to be shared among the voters (P-power in Felsenthal and Machover (2004)). The more powerful the voter in the sense of P-power, the larger the share he or she receives. While this applies to the Shapley-Shubik index, the Penrose-Banzhaf measure is best interpreted as a measure of a voter's influence on voting outcome (I-power).

The **Homogeneity Assumption** is based on the argument that any two votes are independent, conditioned on a realisation p. Consequently, the conditional probability that s of n players vote affirmatively equals $\pi_s(p) = p^s(1-p)^{n-s}$, where $s = 0, \ldots, n$. Under the assumption of a uniform distribution of p, the unconditional probability of observing a voting profile with s affirmative votes is given by

$$\pi_s = \frac{s!(n-s)!}{(n+1)!}. \tag{3}$$

Expressed using the notation of simple voting games with $s = |S|$ and $n = |N|$, and substituted in Eq. 1, produces the familiar expression for the Shapley-Shubik index:

$$\phi(i) = \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \sum_{S \subseteq N \ , \ i \in S} [\mathbf{w}(S) - \mathbf{w}(S \setminus \{i\})].$$

The Homogeneity Assumption thus assumes that the probability of each player voting Yes is drawn from the uniform distribution on $[0, 1]$, and that every player shares the same p. The rationale being that voters follow common standards when evaluating a ballot proposal. Having assigned the common probability of an affirmative vote, voters cast their votes independently. But to an external observer who does not know the value of p, decisions by voting bodies with p close to 0 or 1 would appear highly correlated, as near-unanimous outcomes would be frequent.

Exchangeability. The two models induce different distributions at the level of voting profiles or coalitions. In both cases, however, the probability of a voting profile is invariant to a permutation of votes in the voting profile. The fact that both the Independence Assumption and the Homogeneity Assumption lead to exchangeability among voters allows us to carry out some simplifications during our computational tasks later in this paper. Exchangeability allows us to significantly reduce the number of parameters required to define the joint probability distribution of votes, because in a model with exchangeable votes each vote has the same expected value, and each pair of votes correlates with the same correlation coefficient. The latter fact allows us to unambiguously talk about the overall level of correlation in a stochastic model. For further discussion of exchangeability in the context of power indices, see Ruff and Pukelsheim (2010).

Positive correlation. Homogeneity makes broad coalitions more probable than tight coalitions. Broad coalitions represent voting profiles with a high degree of consensus, and are characterised by a high percentage of zeros or ones in the voting profile. This differential effect on the probability of unequally-sized coalitions is consistent with positive correlation between concordant votes and negative correlation between discordant votes.

Although the Bahadur parametrisation admits negative correlations in principle, our analysis focuses on positive correlations for a number of reasons. Firstly, it is positive correlation that prevails in the data. Indeed, voting data for judicial bodies, such as the U.S. Supreme Court (Kaniovski and Leech 2009) and the Supreme Court of Canada (Heard and Swartz 1998), as well as non-judicial bodies, such as the Council of the European Union (Hayes-Renshaw, van Aken and Wallace 2006) and the United Nations (Newcombe, Ross and Newcombe 1970), show that voting outcomes with a high degree of consensus are common, and that the votes correlate positively. The work of Gelman, Katz and Boscardin (2003) and Gelman, Katz and Bafumi (2004) shows that this phenomenon also arises in general elections.

Secondly, the range of admissible negative correlation for a joint probability distribution to exist is much narrower than in the case of positive correlation. Negative correlation presents an awkward constraint in binary choice situations. The case of perfect negative correlation provides an intuition for this. If one voter votes Yes whenever another votes No, and vice versa, then by virtue of binary choice a third voter cannot be simultaneously discordant with the former two, i.e. three voters cannot be mutually contrarian in a two-way election. Positive correlations do not impose such a constraint, because they reflect common rather than contrarian tendencies. Positive correlations between affirmative votes make all such votes more probable than they would otherwise be if the votes were independent.

Thirdly, and most importantly, negative correlation in the context of simple voting games is at odds with the mindset of the cooperative game theory, as negative correlation clearly indicates the absence of cooperation. All of the above considerations strongly point to the case of positive correlations as the only case relevant for simple voting games.

3 Bahadur's parametrisation

Let v_i be a realisation of a binary random variable V_i, such that: $P(V_i = 1) = p_i$ and $P(V_i = 0) = 1 - p_i$, where $p_i \in (0, 1)$ for all i. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the vector of n realisations, occur with the probability $\pi_{\mathbf{v}}$. Bahadur (1961) obtained the representation of the joint probability distribution of $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ in terms of n marginal probabilities and $2^n - n - 1$ correlation coefficients of all orders.

The Bahadur representation is sufficiently general to define the joint probability distribution of *any* vector of correlated binary variables. The drawback is the large number of parameters it therefore requires. Bahadur proposed truncating the joint probability distribution to second order correlations:

$$\pi_{\mathbf{v}} = \prod_{i=1}^{n} p_i^{v_i}(1-p_i)^{1-v_i}\left(1 + \sum_{1\le i<j\le n} c_{i,j} z_i z_j\right) \text{ for each } i = 1, 2, \ldots, n; \tag{4}$$

where z_i is a realisation of the normalised random variable $Z_i = (V_i - p_i)/\sqrt{p_i(1-p_i)}$. The first factor in the above formula is the probability of $\mathbf{V} = \mathbf{v}$ under independence.

The second order correlation coefficient in Eq. 4 is the Pearson product-moment correlation coefficient between two binary random variables V_i, V_j with $E(V_i) = p_i$, $E(V_j) = p_j$:

$$c_{i,j} \equiv \mathsf{Corr}(V_i, V_j) = \frac{P\{V_i = 1, V_j = 1\} - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}.$$

The Pearson product-moment correlation coefficient is the most widely used measure of stochastic dependence between two random variables. There will be $n(n-1)/2$ such coefficients. Higher order correlation coefficients measure dependence between the general tuples of binary random variables. These higher order correlation coefficients are set to zero in the above equation.

Truncation restricts the range of admissible correlation coefficients. Since $\pi_{\mathbf{v}} \le 1$ by construction, we only need to ensure $\pi_{\mathbf{v}} \ge 0$. Bahadur provided a lower bound on the smallest eigenvalue λ_{min} of the correlation matrix required to ensure $\pi_{\mathbf{v}} \ge 0$ for all $\mathbf{v}$:

$$\lambda_{min} \ge 1 - \frac{2}{\sum_{i=1}^{n} \beta_i}, \text{ where } \beta_i = \max\left\{\frac{p_i}{1-p_i}, \frac{1-p_i}{p_i}\right\} \text{ for each } i = 1, 2, \ldots, n.$$

The above bound is sufficient but not necessary for $\pi_{\mathbf{v}}$ to be a distribution. More precise bounds are available for exchangeable distributions with vanishing higher-order correlations.

In keeping with the work of Straffin, we turn to distributions that observe the exchangeability property. Since in any such distributions $p_i = p$ for all i and $c_{ij} = c$ for all $i < j$, the number of parameters required to define a distribution is reduced to two: p, the marginal probability of a Yes vote, and c, the correlation coefficient between any two such votes.

Bahadur provides the following convenient calculation for the probability of n exchangeable binary random variables, which, in our particular application, translates to the probability of a coalition of size $s = |S|$ forming from a voting assembly of size $n = |N|$, with the common probability of voting in favour of p and a second order correlation coefficient of c:[2]

$$\pi_S = p^{|S|}(1-p)^{(|N|-|S|)}\left\{1 + \frac{c}{2p(1-p)}\left[|S|^2 - |S| + p(|N|-1)(p|N| - 2|S|)\right]\right\}. \tag{5}$$

Bahadur provided exact bounds $[\underline{c}, \overline{c}]$ on the second order correlation coefficient c:

$$-\frac{2(1-p)}{n(n-1)p} \leq c \leq \frac{2p(1-p)}{(n-1)p(1-p) + 0.25 - \gamma}, \tag{6}$$

where $\gamma = \min_{0\leq s\leq n}\{[s-(n-1)p-0.5]^2\} \leq 0.25$ and $s = |S|$. The bound is tighter for negative correlation than it is for positive correlation, as $\overline{c} \sim O(n^{-1})$, but $\underline{c} \sim O(n^{-2})$. Kaniovski and Zaigraev (2011) show that c can be at most $\frac{1}{n-1}$ for $p \approx 0$ or $p \approx 1$, and at most $\frac{2}{n-1}$ for $p \approx 0.5$. The fact that $\underline{c}, \overline{c} \to 0$ as $n \to \infty$ implies that in a large assembly each pair of votes may be only weakly dependent.

Figure 1: Bounds on c for $n = 3, 4, 5, 6$.

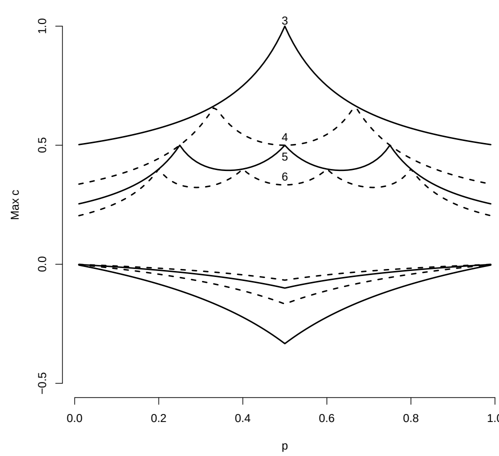


Figure 1 illustrates the bounds on c for $n = 3, 4, 5, 6$. The bounds tighten as n increases or as p approaches 0 or 1 for positive as well as for negative correlations. The lower bound is much tighter for all parameter sets, but especially so when p is close to 0 or 1. Table 1 provides the bounds for a sequence of values of p (starting from $p \geq 0.5$ due to the symmetry about $p = 0.5$).

[2]George and Bowman (1995) provide a more compact representation of the probabilities for exchangeable binary random variables. However, the George and Bowman parametrisation is unsuitable for our simulations because it is not formulated in terms of marginals and correlation coefficients. Kaniovski (2008) formulates a quadratic optimization problem for finding a distribution with given marginals and correlations. For a recent survey of algorithms for generating correlated binary random variables, see Preisser Jr. and Qaqish (2012).

Table 1: Bounds on c for $n = 3, 4, 5, 6$ for $p \geq 0.5$.

	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$
			Lower Bound		
$n = 3$	-0.333	-0.222	-0.143	-0.083	-0.037
$n = 4$	-0.167	-0.111	-0.071	-0.042	-0.019
$n = 5$	-0.100	-0.067	-0.043	-0.025	-0.011
$n = 6$	-0.067	-0.044	-0.029	-0.017	-0.007
			Upper Bound		
$n = 3$	1.000	0.750	0.636	0.571	0.529
$n = 4$	0.500	0.545	0.583	0.444	0.375
$n = 5$	0.500	0.400	0.420	0.400	0.300
$n = 6$	0.333	0.400	0.323	0.400	0.257

Let us illustrate the use of Bahadur's parameterisation in an analysis of voting power. Table 2 shows every possible voting coalition that can be formed with three players. For each coalition, we list the probability of it occurring under the Independence Assumption, the Homogeneity Assumption and three distributions on based parsimonious versions of Bahadur's parameterisation with different values for the parameters p and c. The Bahadur distributions produce patterns of probabilities more varied than those found in the stochastic models by Straffin. We can see:

i). Increasing positive correlation increases the probability of occurrence of broad coalitions;

ii). Increasing p shifts the probabilities toward coalitions with a high percentage of ones;

iii). An additional increase in positive correlation negates some of this shift due to an increase in the probability of occurrence of all broad coalitions, including those with a high percentage of zeros;

iv). That coalitions with the same number of ones are equally likely to occur (Exchangeability).

Table 2: Three examples of Bahadur distributions for $n = 3$.

Votes			Independence	Homogeneity		Bahadur	
v_1	v_2	v_3	(Penrose-Banzhaf)	(Shapley-Shubik)	$p = 0.5$, $c = 0.2$	$p = 0.7$, $c = 0$	$p = 0.7$, $c = 0.2$
1	1	1	0.125	0.250	0.2	0.343	0.431
1	1	0	0.125	0.083	0.1	0.147	0.101
1	0	1	0.125	0.083	0.1	0.147	0.101
1	0	0	0.125	0.083	0.1	0.063	0.067
0	1	1	0.125	0.083	0.1	0.147	0.101
0	1	0	0.125	0.083	0.1	0.063	0.067
0	0	1	0.125	0.083	0.1	0.063	0.067
0	0	0	0.125	0.250	0.2	0.027	0.065

3.1 Bahadur Parameterised Voting Power

Everything is now in place to construct a method for measuring voting power in games where the joint probability distribution of the voters can be modelled by just two parameters p and c.

Recall that we defined in (1) that the power of a voter i is given by:

$$\sum_{T\subset N\ ,\ i\notin T} \pi_T[\mathbf{w}(T\cup\{i\})-\mathbf{w}(T)] \quad + \sum_{S\subseteq N\ ,\ i\in S} \pi_S[\mathbf{w}(S)-\mathbf{w}(S\setminus\{i\})], \tag{7}$$

where π_T and π_S are the probability of occurrence of coalitions T and S, such that $T = S\setminus\{i\}$. We only need replace π_T and π_S with the truncated second order Bahadur probability model given in (5) to complete our method:

$$\sum_{T\subset N\ ,\ i\notin T} p^{|T|}(1-p)^{(|N|-|T|)}\left\{1+\frac{c}{2p(1-p)}\left[|T|^2-|T|+p(|N|-1)(p|N|-2|T|)\right]\right\}[\mathbf{w}(T\cup\{i\})-\mathbf{w}(T)]$$

$+$

$$\sum_{S\subseteq N\ ,\ i\in S} p^{|S|}(1-p)^{(|N|-|S|)}\left\{1+\frac{c}{2p(1-p)}\left[|S|^2-|S|+p(|N|-1)(p|N|-2|S|)\right]\right\}[\mathbf{w}(S)-\mathbf{w}(S\setminus\{i\})],$$

where p is the probability of voting yes, c is the second order correlation coefficient, and $T = S\setminus\{i\}$.

4 Analysing Games With Correlated Voters Using the Standard Measures

How do the two most widely used voting power measures, Penrose-Banzhaf and Shapley-Shubik, compare when used to analyse correlated voting? In order for the comparison to be meaningful it is important to examine a large number of different games, with a large number of different correlated probability models. The approach we adopted was to examine *all* simple voting games with up to a given (small) number of players. Since simple voting games include weighted voting games as a subclass, our results indicate what can be expected within weighted voting games too. Creating a large number of different probability models (stochastic models) was accomplished by varying values of p and c on a fine numerical grid. The resulting number of different games and stochastic models is given in Table 3. The number of stochastic models decreased with increasing n, as a result of the possible values of c decreasing with increasing n (Figure 1).

For every game, every stochastic model, and every player, we calculated the voting power using the Bahadur parameterisation, and compared it with the voting power calculated using the Penrose-Banzhaf measure and the Shapley-Shubik index. In other words, we calculated the difference in performance between the Bahadur based voting power measure and the standard techniques when used in games with correlated voters that can be described (or approximated) with a second order correlation coefficient. We averaged this difference for each standard technique to obtain a single quantity for each game, which we call the Game Mean Difference (GMD).

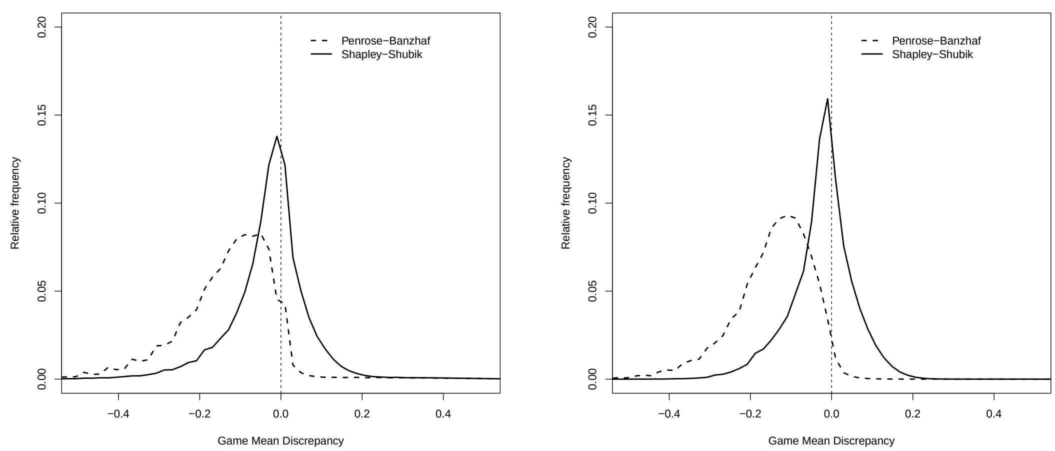
Table 3 reports the *average* GMD for all SVGs with n players, where the average is taken for all games; Figure 2 shows the average GMD for $n = 5$ and $n = 6$. A negative average GMD implies that, on average, any voting power analysis carried out using the standard techniques will apportion too much power to the players. A positive GMD implies that the measure tends to underestimate power. We can see that, on average, in games where the probability of a Yes vote deviates from 0.5 and the votes are positively correlated, i). both the Penrose-Banzhaf measure and Shapley-Shubik index overestimate power, and;

ii). the Shapley-Shubik index provides a better estimate of voting power.

The main implication of these results are that, despite the Penrose-Banzhaf and Shapley-Shubik measures being valid measures of *a priori* voting power and thus useful for evaluating the rules at the constitutional stage of a voting body, they are, in general, poor measures of the actual probability of being decisive in games with correlations between the voters, or with non-equiprobable voting. Neither measure can be used to forecast how frequent a voter will be decisive with any reasonable degree of certainty.

Table 3: Mean GMD ($c \geq 0$).

n	# Games	# Stoch. Models per Game	Penrose-Bazhaf	Shapley-Shubik
2	5	10,000	-0.002	-0.002
3	19	6,518	-0.048	-0.022
4	167	4,883	-0.104	-0.037
5	7,580	3,943	-0.139	-0.040
6	7,828,353	3,304	-0.154	-0.034

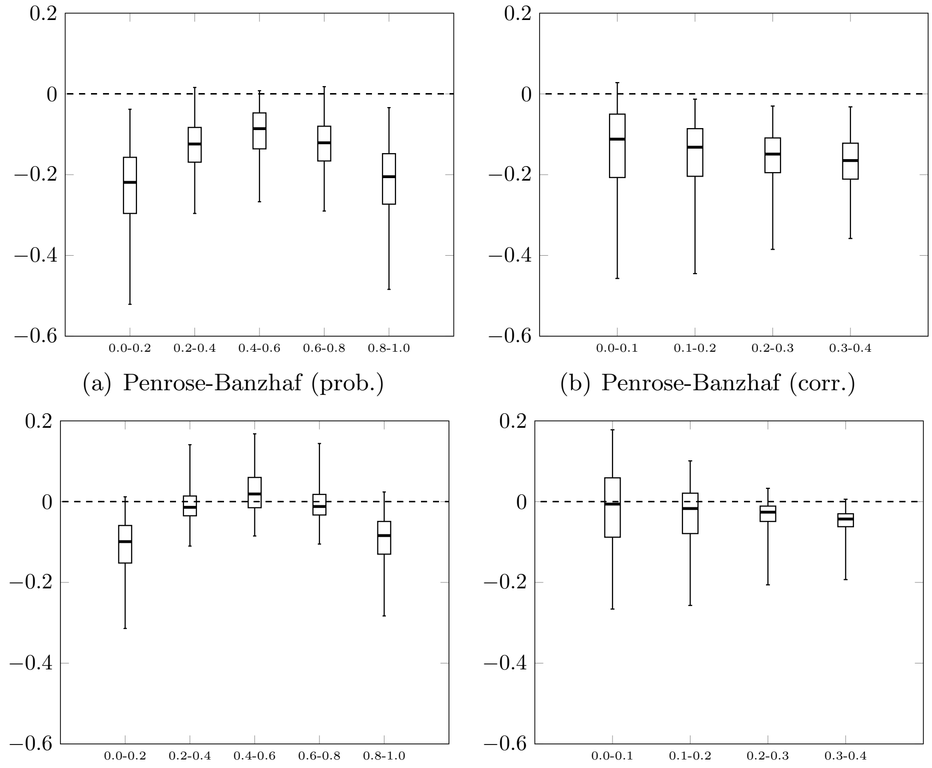


(a) $n = 5$ (b) $n = 6$

Figure 2: The average GMD for $n = 5$ and $n = 6$ for $c \geq 0$.

To take a closer look at how the GMD varies with p and c, we focus on simple voting games with six players, as this gives us the largest number of comparable voting games. A boxplot is helpful for detecting asymmetries in a distribution. Figure 3 shows boxplots with the principal quartiles of the GMD for ranges of p and c. The bar in the middle represents the median or the 50% quantile. The top whisker ranges from the 99% quantile to the 75% quantile, while the bottom whisker ranges from the 25% quantile to the 1% quantile.

Looking at the panels in the left column of Figure 3, we observe that both measures improve as they approach the midrange of $p \in [0.4, 0.6]$, although on closer inspection it appears that the



(a) Penrose-Banzhaf (prob.)

(b) Penrose-Banzhaf (corr.)

(c) Shapley-Shubik (prob.)

(d) Shapley-Shubik (corr.)

Figure 3: The average GMD by intervals of p and c ($c \geq 0$) for $n = 6$.

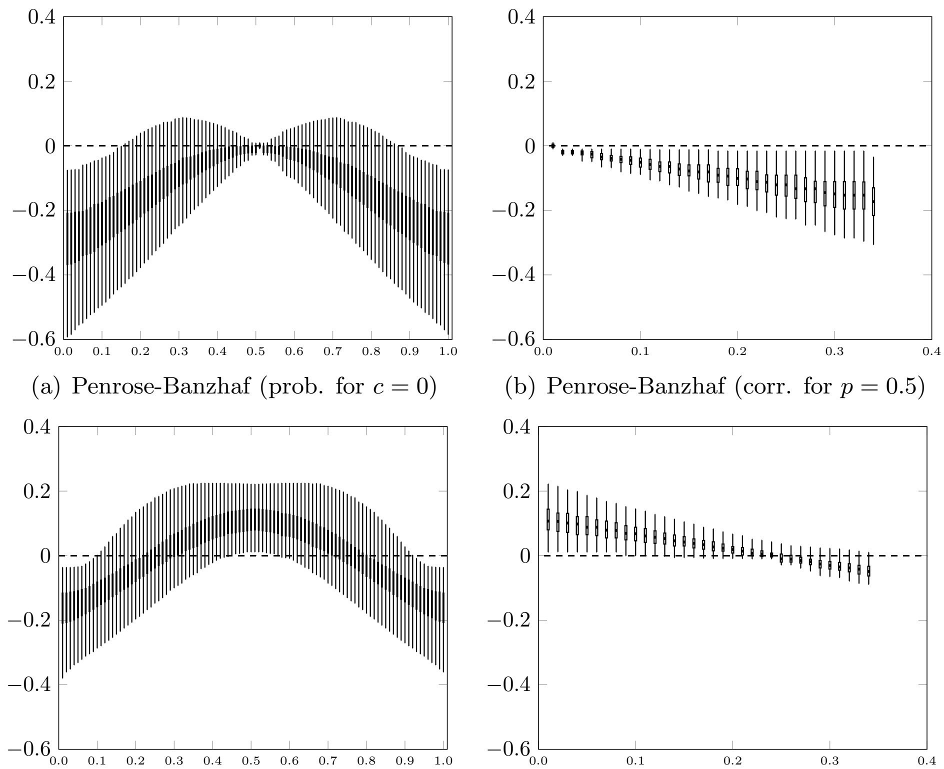
Shapley-Shubik index performs better at small distances away from $p = 0.5$. For all probability intervals, the median GMD shown by the horizontal bars in the middle is closer to zero for the Shapley-Shubik index than it is for the Penrose-Banzhaf measure.

The right column of Figure 3 tells us that Penrose-Banzhaf performs best for low correlation. The Shapley-Shubik index exhibits slightly more interesting behaviour. While the average GMD for the Shapley-Shubik index is negative, the index may underestimate rather than overestimate power for distributions with $p \approx 0.5$, or $c \approx 0$.

Like Penrose-Banzhaf, the median GMD of the Shapley-Shubik index worsens slightly with increasing correlation; however there is a corresponding decrease in the variance of the differences. But while the accuracy of the Shapley-Shubik index decreases slightly, its spread also decreases, so that the Shapley-Shubik index appears superior for all considered values of c.

The results so far tell us that the effect of variations in p on the average GMD is larger than the effect of variations in c. This is consistent with the results in Good and Mayer (1975), Chamberlain and Rothschild (1981) and Grofman (1981) for the Penrose-Banzhaf measure in symmetrically weighted simple majority games (general elections). These earlier studies make clear that the probability of being critical changes considerably when the votes are not equiprobable. This is also consistent with the more recent analysis in Kaniovski (2008) for the Penrose-Banzhaf measure, which includes the case of correlated votes.

In order to try and isolate how changes in p and c affect the GMD, we now go on to examine in greater detail some special cases. To determine how changes in probability affect the GMD, we first examine only those stochastic models that used a value of $c = 0$. Then, we will examine only those models that used a value of $p = 0.5$, to determine how changes in correlation affect the GMD. The distributions of GMD are shown in Figure 4. The left panel shows the distributions for independent votes ($c = 0$), and the right panel shows the equiprobable votes ($p = 0.5$). Note, the range of admissible values of c is restricted by Equation (6), and according to the Bahadur parametrisation in Equation (5), both c and p remain closely linked, making a clean separation of effects impossible.



(a) Penrose-Banzhaf (prob. for $c = 0$) (b) Penrose-Banzhaf (corr. for $p = 0.5$)

(c) Shapley-Shubik (prob. for $c = 0$) (d) Shapley-Shubik (corr. for $p = 0.5$)

Figure 4: The average GMD by intervals of p and c ($c \geq 0$) for $n = 6$.

The left panel shows that for both Penrose-Banzhaf and Shapley-Shubik as p moves away from 0.5 the GMD becomes increasingly negative. But, more interestingly, we see that for $p = 0.5$ the Shapley-Shubik index consistently underestimates voting power. Examining the right panel we see that as c increases the GMD becomes increasingly negative for both Penrose-Banzhaf and Shapley-Shubik, it is almost a linear relationship. Focusing on the Shapley-Shubik index, we note that it starts off underestimating voting power when $c = 0$, and then continues to become more accurate as it approaches $c = 0.3$ from where it starts to overestimate voting power. This could be taken in some ways as an experimental verification of a result noted in (Comment 6.3.16, Felsenthal and Machover 1998), where it is was shown that the Shapley-Shubik index implies a correlation coefficient of 1/3.

We can summarise these results as follows:

i). In general, moving away from equiprobable votes tends to make the power indices overestimate voting power;

ii). Increasing correlation tends to make the power indices overestimate voting power;

iii). For almost all situations where $p = 0.5$ and $c = 0$, the Shapley-Shubik index will underestimate voting power.

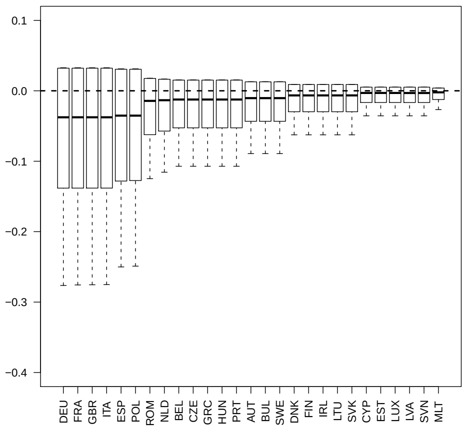
5 Voting power in the Council of the European Union

Our simulations for small voting games show that the Shapley-Shubik index is closer to the true probability of criticality when the Yes and No votes are neither equally probable nor independent. The following calculations show the magnitude of the difference between the Penrose-Banzhaf and Shapley-Shubik measures for the Council and their counterparts based on more general distributions. The need to base power calculations on more flexible and more empirically informed distributions of votes has been repeatedly voiced in the literature (e.g., in Garrett and Tsebelis 1999a, Garrett and Tsebelis 1999b).

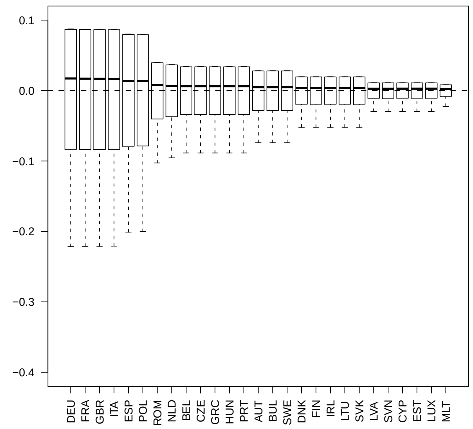
At the time of writing this paper, the EU comprised 27 member states. According to the Treaty of Nice – signed on the February 26, 2001, and in force since February 1, 2003 – qualified majority voting in the Council requires at least 255 out of the total of 345 votes to pass a motion. In addition to a qualified majority of votes, a Council decision requires support from more than fourteen member states to pass, as well as support from states representing at least 62 percent of the EU population. These three elements constitute the triple majority requirement of the Treaty of Nice. The Treaty of Lisbon sidestepped the potentially contentious issue of allocating votes among the member states in favour of a double-majority voting system based on the number of countries and populations only. Under the double-majority rule of the Treaty of Lisbon – signed on December 13, 2007 – a motion will pass if at least fifteen member states representing at least sixty-five percent of the EU population support the motion, unless it is opposed by at least four member states. The above voting rules apply when the Council acts on a proposal by the European Commission, as would be the case under the ordinary legislative procedure, formerly the co-decision procedure. This is the main procedure by which legislations are passed.

In our power calculation for the Treaty of Nice we used the Eurostat population data for 2010. Calculations for the Lisbon Treaty used population projections for 2060, as in Felsenthal and Machover (2009) and Kóczy (2012). Figure 5 summarises the distribution of difference and Table A.1 reports the means over stochastic scenarios. Unlike in the presentation of the simulations for all small games, we do not average over the players in a game because we have a single game. We therefore present the simulation results by players, first the total difference and then the difference due to p deviating from 0.5 for $c = 0$, and c deviating from 0 for $p = 0.5$.

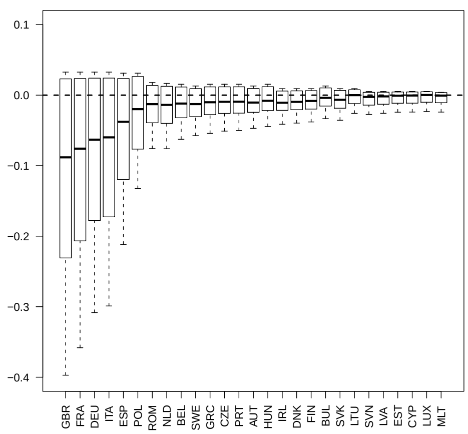
The countries in Figure 5 are arranged in the descending order of their Penrose-Banzhaf and Shapley-Shubik powers. Figure 5 confirms the results of our simulations for small games. The difference in the Shapley-Shubik index is considerably lower than the difference in the Penrose-Banzhaf index (see the means in Table A.1). But Figure 5 also shows that this difference is larger when a country is more powerful. In fact, the average difference over all stochastic scenarios is almost perfectly negatively correlated with voting power according to the traditional measures, with the correlation coefficients between differences and powers ranging from -1 to -0.88. This is not surprising, given that, the more powerful a voter is, the more numerous, on average, the coalitions in which he is critical will be. For a powerful voter, differences accumulate over a larger number of coalitions, resulting in a larger average differences. The differences under the Treaty of Lisbon are larger because there are more winning coalitions and, hence, more coalitions in which a voter can be critical. The share of winning coalitions in the total number of coalitions



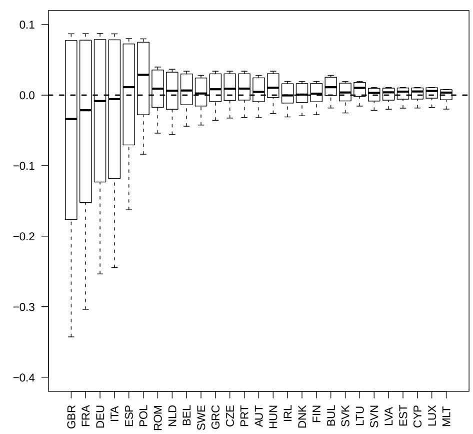
(a) Penrose-Banzhaf (Nice)



(b) Shapley-Shubik (Nice)



(c) Penrose-Banzhaf (Lisbon)

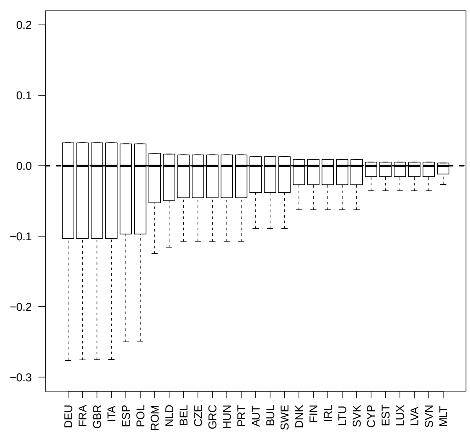


(d) Shapley-Shubik (Lisbon)

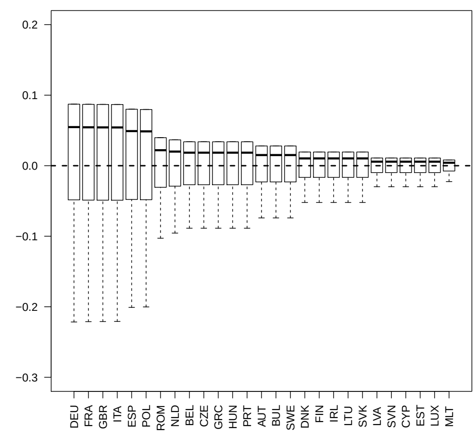
Figure 5: The average difference in the Council of the European Union.

is known as Coleman's power of a collectivity to act (Coleman 1971). This efficiency will rise significantly once the Lisbon voting rules come into effect, as the share of winning coalitions will increase from 0.020 under the Treaty of Nice to 0.127 under the Treaty of Lisbon. This increase in the ability to reach a decision has been analysed in König and Bräuninger (2000), Hosli and Machover (2004) and Doležel (2011).

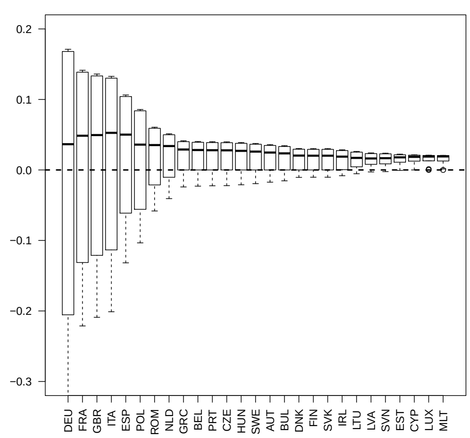
Figures 6 and 7 look at the differences in the special cases $c = 0$ and $p = 0.5$. Figure 7 shows that in this particular game, the bias due to the probability of an affirmative vote deviating from 0.5 is likely to be negative for the Penrose-Banzhaf measure, but can be positive or negative for the Shapley-Shubik index. This is consistent with the overall differences for this game in Figure 5, and the results for all small games in Figure 4 presented in the previous section. The bias in the Shapley-Shubik index is likely to be positive, which is consistent with what we saw in Figures 4 and 5. Looking at Figure 7, we note that the bias due to correlation is much smaller in magnitude than the bias due to the probability of an affirmative vote deviating from $p = 0.5$, which is also consistent with the previous results. One interesting point to note is that the difference distributions for the Lisbon Treaty appear to have a higher dispersion than the difference distributions under the Treaty of Nice. This finding is not surprising given that countries generally have many more swings under the Lisbon voting rules than under the Nice rules, as there are many more winning coalitions under the Lisbon rules than under the Nice rules. The higher number of swings may lead to accumulation of difference under different parameter values (p and c) when the differences tend to have the same sign.



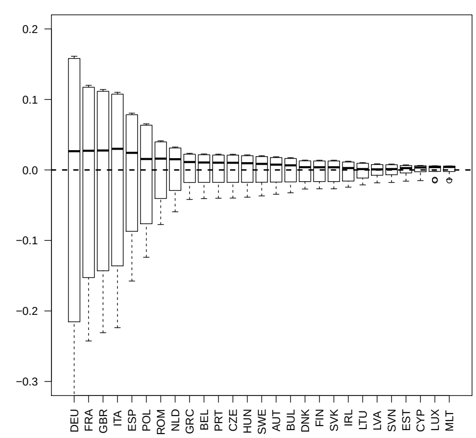
(a) Penrose-Banzhaf (Nice)



(b) Shapley-Shubik (Nice)

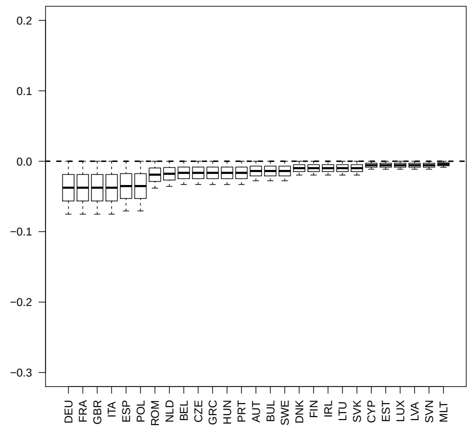


(c) Penrose-Banzhaf (Lisbon)

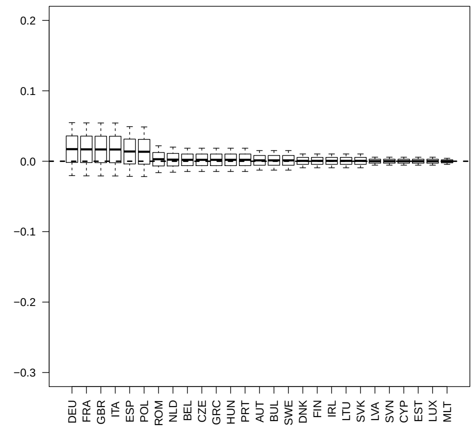


(d) Shapley-Shubik (Lisbon)

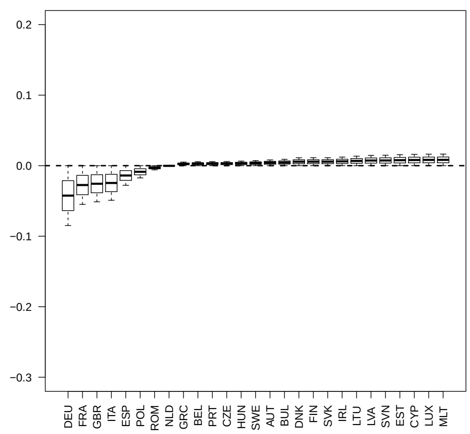
Figure 6: The average difference in the Council of the European Union for $c = 0$.



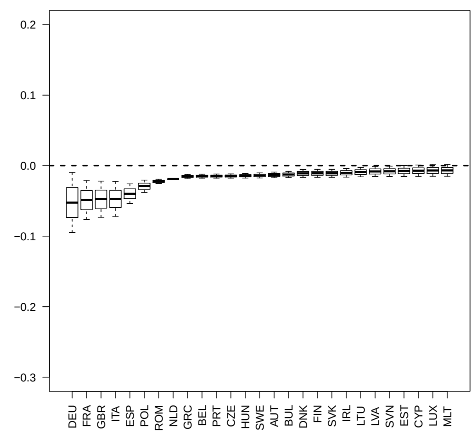
(a) Penrose-Banzhaf (Nice)



(b) Shapley-Shubik (Nice)



(c) Penrose-Banzhaf (Lisbon)



(d) Shapley-Shubik (Lisbon)

Figure 7: The average difference in the Council of the European Union for $p = 0.5$.

6 Concluding remarks

This paper presented the Bahadur parameterisation as a way of both modelling correlated votes and a tool to calculate voting power in games with correlated voters. Armed with a technique for calculating voting power in games with correlations we set out to answer a related question. We knew that the vast majority of voting power analysis makes use of either the Penrose-Banzhaf measure or the Shapley-Shubik index. We wanted to understand just how much difference could arise if you used the standard techniques to analyse games with correlations between the voters. Our work showed that the Shapley-Shubik index consistently performed better than the Penrose-Banzhaf measure in games with correlations.

However, even the Shapley-Shubik index cannot forecast how frequently a voter will be decisive. Predicting this would require a more complex model that allows varying probabilities of affirmative votes and correlation coefficients between them. While the full Bahadur parametrisation can, in principle, deliver such a model, its informational requirements are prohibitively high. We believe the truncated Bahadur model presented in this paper is a step in the right direction.

Do our findings invalidate the Penrose-Banzhaf measure of voting power? No, they do not. We believe that Penrose-Banzhaf is the right tool when one wishes to presume maximum freedom of choice for the voter and maximum freedom of choice for the voting assembly. As such, it is ideally suited for designing fair constitutional assemblies.

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Table A.1: Mean difference in the Council of the European Union.

	Nice Treaty		Lisbon Treaty	
	Penrose-Banzhaf	Shapley-Shubik	Penrose-Banzhaf	Shapley-Shubik
AUT	-0.017	-0.002	-0.009	0.007
BEL	-0.021	-0.002	-0.013	0.005
BUL	-0.017	-0.002	-0.003	0.012
CYP	-0.007	-0.001	-0.004	0.002
CZE	-0.021	-0.002	-0.009	0.010
DEU	-0.061	-0.006	-0.088	-0.033
DNK	-0.012	-0.001	-0.008	0.002
ESP	-0.055	-0.006	-0.059	-0.010
EST	-0.007	-0.001	-0.004	0.002
FIN	-0.012	-0.001	-0.007	0.003
FRA	-0.060	-0.006	-0.105	-0.050
GBR	-0.060	-0.006	-0.117	-0.063
GRC	-0.021	-0.002	-0.010	0.008
HUN	-0.021	-0.002	-0.006	0.012
IRL	-0.012	-0.001	-0.009	0.002
ITA	-0.060	-0.006	-0.085	-0.031
LTU	-0.012	-0.001	-0.003	0.008
LUX	-0.007	-0.001	-0.003	0.002
LVA	-0.007	-0.001	-0.005	0.001
MLT	-0.005	-0.001	-0.004	0.000
NLD	-0.023	-0.003	-0.019	0.002
POL	-0.054	-0.006	-0.028	0.021
PRT	-0.021	-0.002	-0.008	0.010
ROM	-0.025	-0.003	-0.018	0.004
SVK	-0.012	-0.001	-0.006	0.004
SVN	-0.007	-0.001	-0.005	0.000
SWE	-0.017	-0.002	-0.013	0.002